Study of Some Holographic Superconductors

A Thesis Submitted

To

Sikkim University

In Partial Fulfilment of the Requirement for the
Degree of Doctor of Philosophy

By

Chandrima Paul
Department of Physics,
School of Physical Sciences

May 2019

DECLARATION

I declare that the thesis entitled " **Study of Some Holographic Superconductors**" submitted by me for the award of **Doctor of Philosophy in Physics** of Sikkim University is my original work. The content of this thesis is based on the works which I have performed myself. This thesis has not been submitted for any other degree to any other University. The content of this Ph.D Thesis has been subjected to the Anti-Plagiarism Software (Urkund) and was found satisfactory.

Date: 14.05.2019

Chandrima Paul
(Chandrima Paul)

Roll No : 12PDPY03

Reg. No : 13/PD/PY/02

Recommended that the Thesis be placed before the Examiners for evaluation.

14-05-2019

(Dr. Subir Mukhopadhyay)

Ph.D Supervisor

6 माइल, सामदुर, तादोंग -737102
गंगटोक, सिक्किम, भारत
फोन-03592-251212, 251415, 251656
टेलीफैक्स -251067
वेबसाइट - www.cus.ac.in

6th Mile, Samdur, Tadong -737102
Gangtok, Sikkim, India
Ph. 03592-251212, 251415, 251656
Telefax: 251067
Website: www.cus.ac.in

सिक्किम विश्वविद्यालय
SIKKIM UNIVERSITY
(भारत के संसद के अधिनियम द्वारा वर्ष 2007 में स्थापित और नैक (एनएएसी) द्वारा वर्ष 2015 में प्रत्यायित केंद्रीय विश्वविद्यालय)
(A central university established by an Act of Parliament of India in 2007 and accredited by NAAC in 2015)

CERTIFICATE

This is to certify that the thesis entitled "**Study of Some Holographic Superconductors**" submitted to Sikkim University in partial fulfillment of the requirements for the degree of **Doctor of Philosophy in Physics** embodies the result of *bona fide* research work carried out by **Chandrima Paul** under my guidance and supervision. No part of the thesis has been submitted for any other degree, diploma, associate-ship, fellowship.

All the assistance and help received during the course of the investigation have been duly acknowledged by her.

Place : Gangtok, Sikkim

Date: 14-05-2019

(Dr. Subir Mukhopadhyay)

Ph.D Supervisor

Department of Physics

School of Physical Sciences

PLAGIARISM CHECK CERTIFICATE

This is to certify that plagiarism check has been carried out for the following Ph.D Thesis with the help of URKUND software and the result is within the permissible limit decided by the University.

" Study of Some Holographic Superconductors"

Submitted by **Chandrima Paul** under the supervision of **Dr. Subir Mukhopadhyay** of the Department of Physics, School of Physical Sciences, Sikkim University, Gangtok, India.

Chandrima Paul 14.05.2019
Signature of the Candidate

14.05.2019
Countersigned by Ph.D Supervisor

Vetted by Librarian 20/5/19

Dedication

Dedicated in the Memory of my beloved brother

Rupak Kumar Paul

Acknowledgement

I express my deep sense of gratitude to my respected supervisor Dr. Subir Mukhopadhyay under whose able guidance my PhD thesis has been completed. My cordial thanks are due to most respected Prof. Tanka Bahadur Subba, ex-Vice-Chancellor of Sikkim University, whose advice and kind help has inspired me so much for the successful completion of my work. I am also grateful to Dr. Amitabha Bhattacharya, ex-Head of the department of Physics for his kind cooperation and timely advice to arrange the submission of my thesis. My thanks are also due to all the teachers of the department who were so kind enough to make the subject lucid through their patient lectures.

I am also grateful to my friends at Physics department and Microbiology department for their cooperation and giving me constant inspiration. My thanks are due to Mr. Bikram Thapa and Mr. N.B. Subba of Physics Laboratory for their timely assistance. I am also indebted to the system administrators of the University for making cooperation in installing Linux in my laptop and other technical helps pertaining to electronic affairs.

I am also greatly indebted to the Dean of Students Welfare, Dr. Manivannan for his sincere help in solving my various problems. I would like to thank all my hostel wardens, particularly to Dr. Nidhi Saxena for her kind help.

My sincere thanks are due to Dr. Rakesh Basnet, Provost, RPJC girls' hostel and to the Registrar of Sikkim University, Mr. T.K.Kaul, for their kind cooperation. during my visit to Kyoto University, Japan, for participating in the "Strings, gravity and Cosmology" workshop, September 2017.

I am grateful to my parents and aunt for supporting me continuously.

Financial assistance rendered by U.G.C is gratefully acknowledged. I thank Sikkim University for providing me the opportunity for doing Ph.D.

Synopsis

According to the AdS/CFT correspondence, type IIB string theory on $AdS_5 \times S^5$ is dual to the four-dimensional N = 4, SU(N) gauge theory on the boundary. In the weak coupling limit and large curvature, the string theory can be approximated by the gravity description and so gravity theory on (d+1) dimensional AdS spacetime corresponds to gauge theory living on the d-dimensional boundary. Since this duality connects the weakly coupled gauge theory to strongly coupled gravity theory and vice versa, this connection provides a way to describe strongly coupled problems of the gauge theory in the set up of weakly coupled gravity theory, which can be analysed more easily. Soon it transpires that this can be extended to other field theories as well. In fact the radial direction of the AdS space turns out to corresponds to the energy scale of the dual field theory in the sense that theory living on d-dimensional slices at definite values of radial variable in the bulk corresponds to field theory at a definite energy scale. This duality has been widely applied to the context of various strongly coupled theories, such as quark-gluon-plasma or strongly correlated systems in condensed matter theories, among others.

In this thesis we will discuss such applications in the context of superconductors. Being a finite density, finite temperature systems, dual of such systems can be realised by charged black holes in gravity theory. It turns out that black holes in presence of matter fields develop instability and below a critical temperature, configuration with condensation of matter fields becomes thermodynamically more stable, which translates in field theory side into transition to a superconducting phase, more commonly called as holographic superconductor. The holographic superconductors may correspond to condensation of scalar field (s-wave), vector field (p-wave) or tensor field (d-wave). In general, the superconductors are spatially homogeneous but in condensed matter there are ample examples of spatially modulated superconducting phases as well.

We are mostly interested in such superconducting states with spatial modulation. We are motivated by the work [1], where authors have studied two different phases, namely metallic and insulating phase. Indeed, superconductors admit insulating phase characterised by antiferromagnetic properties and it has been suggested [4] that such antiferromagnetic property may correspond to condensation of scalar field, which are in adjoint of SU(2). In the spirit of that suggestion, we have considered a model involves gravity coupled with $SU(2) \times U(1)$ gauge theory with a scalar field in adjoint of SU(2) along with a Chern Simons term. The Chern-Simons term plays a crucial role in condensation of the vector field. One of the solution of our theory is RN-AdS black hole, which we have identified with the normal phase. Below critical temperature, it gives rise to s wave phase through condensation of scalar, p wave phase through condensation of $SU(2)$ gauge field, or s+p wave phase through condensation of both. We have studied the variation of free energy along with temperature for all the phases. We found for a given temperature s wave phase has least free energy, p wave phase has even higher and s+p wave phase has highest. So we may conclude that in the thermodynamically most preferred state the scalar field condenses. All the phases make transition to normal phase (i.e the phase without any condensate) at critical temperature and all the phase transition are second order in nature. For p wave phase, from the study of free energy vs temperature for various pitches we observe that there is a value of pitch k for which free energy is minimum. We have also studied ac conductivities of this model.

The abovementioned study was done within the context of Einstein gravity which corresponds to a large N limit and superconducting phase corresponds to spontaneous breakdown of a $U(1)$ symmetry. A similar feature appears in (2+1) dimensional theory as well and in the light of Mermin-Wagner theorem it suggests that the fluctuations may get suppressed at large N limit. In order to study this scenario as one deviates from large N limit, we consider further corrections coming from the higher derivative terms. To be precise, we include Gauss Bonnet terms to the action of the gravitational system. Similar to the Einstein gravity, this also admits black hole solution as well as condensation of matter fields. We have studied the behaviour of the free energy and the condensates with variation of temperature in this model. We have found indeed the higher derivative correction suppresses the condensation i.e the critical temperature decreases with the increase of the strength of higher curvature correction. We have also studied AdS soliton solutions into which the black hole may decay on variation of chemical potential. Such transition happens at zero temperature, and from the consideration of the behaviour of the free energy on variation of chemical potential we find that the critical chemical potential increases with the increase of the strength of higher curvature correction.

As mentioned above, since the radial direction in AdS space corresponds to energy scale of the boundary theory, it provides a convenient set up for study of renormalisation group flow and in the case of gauge/gravity duality it is called as holographic RG flow, where in the gravity theory one considers flow of the space time fields with radial direction of AdS space. In general, in the context of RG flow one is interested in fixed points, which usually refer to those points on energy scale where β function vanishes. Verlinde et. al. showed [2, 3] in gravity theory one can construct similar β functions as well using holographic RG flow and the point where the β function vanishes are associated with fixed points. Thus different phases arised in condensed matter system can be identified as the fixed point of holographic RG flow.

We have applied holographic RG flow in our model and have identified two different fixed points. One fixed point arises at vanishing v.e.v of scalar field, while for the other, scalar field v.e.v lies on a circle. The model admits RN AdS black hole solution for vanishing of scalar field and $SU(2)$ field and it may be noted that [1] the near horizon geometry of RN AdS black hole (along with some deformation) corresponds to metallic phase. The other phase corresponds to breaking of $SU(2) \to U(1)$ and expected to be insulating. It has been suggested that condensation of adjoint $SU(2)$ scalar field corresponds to antiferromagnetic phase [4]. So this study seems to connect metallic and insulating antiferromagnetic phases through holographic RG flow.

Another interesting arena is the strange metal phases which shows anomalous scaling of transport coefficient with temperature. For example, resistivity shows a linear temperature dependence for these phases, which is different from the behaviour predicted by Fermi liquid theory. It has been suggested that such anomalous behaviour may holographically realised through hyperscaling violating asymptotically Lifshitz theories [8]. These theories show different scaling for time and space coordinates and the duality proposed in the context of asymptotically AdS spacetime has been generalised to include other asymptotic geometries. Usual approach to study such transport coefficients is to do near horizon analysis as done in [5–7]. This method, although quite efficient, does not accommodate different boundary conditions and the boundary operators cannot be identified. Instead, we have chosen a different approach proposed in [9] where one can accommodate different boundary conditions. We consider a dyonic black hole solution as background and considering the equations of motion of linear fluctuations in this we study the direct conductivity, Seebeck coefficients, Hall angle etc. The temperature dependence turn out to be quite involved and we identified different scaling regime to obtain a simplified power law behaviour for different ranges of parameters. To summarise we have studied differnt aspects of holographic superconductors, which shows a quite rich and complex phase structures. Incorporation of further ingredients and generalisation of the models considered here may provide better agreement with realistic phenomena in condensed matter theories and may deepen our understanding of holography as well.

Bibliography

[1] Aristomenis Donos, Sean A. Hartnoll, "Metal-insulator transition in holography," Nature Phys. **9**,(2013)649-655 doi: 10.1038/nphys2701 [arXiv:1212.2998[hep-th]].

[2] Jan de Boer, "The Holographic Renormalization Group," Fortsch. Phys. **49**,(2001)339-358 doi: 10.1002/1521-3978(200105)49:4/6¡339::AID-PROP339¿3.0.CO;2-A [arXiv:0101026 [hep-th]].

[3] J. de Boer, E. Verlinde, H. Verlinde, "On the Holographic Renormalization Group," JHEP 0008 **003**,(2000) doi: 10.1088/1126-6708/2000/08/003 [arXiv:9912012 [hep-th]].

[4] N. Iqbal, H. Liu, M. Mezei and Q. Si, Phys. Rev. D **82**, 045002 (2010) doi:10.1103/PhysRevD.82.045002 [arXiv:1003.0010 [hep-th]].

[5] Aristomenis Donos, Jerome P. Gauntlett, "Thermoelectric DC conductivities from black hole horizons," JHEP **1411**, 081 (2014) doi: 10.1007/JHEP11(2014)081 [arXiv:1406.4742 [hep-th]].

[6] Aristomenis Donos, Jerome P. Gauntlett, "Navier-Stokes Equations on Black Hole Horizons and DC Thermoelectric Conductivity," Phys.Rev. D92 (2015) **12**, 121901 doi: 10.1103/PhysRevD.92.121901 [arXiv:1506.01360 [hep-th]].

[7] Elliot Banks, Aristomenis Donos, Jerome P. Gauntlett, "Thermoelectric DC conductivities and Stokes flows on black hole horizons," JHEP **1510**, 103 (2015) doi: 10.1007/JHEP10(2015)103 [arXiv:1507.00234 [hep-th]].

[8] Andreas Karch "Conductivities for Hyperscaling Violating Geometries" JHEP **1406** (2014) 140 [arXiv:1405.2926 [hep-th]]

[9] S. Cremonini, M. Cvetic and I. Papadimitriou, JHEP **1804**, 099 (2018) doi:10.1007/JHEP04(2018)099 [arXiv:1801.04284 [hep-th]].

List of publications

1. Subir Mukhopadhyay,Chandrima Paul, *Phases in holographic helical superconductor*,International Journal of Modern Physics A Vol. 32, No. 13 (2017) 1750064,DOI: 10.1142/S0217751X17500646

2. Subir Mukhopadhyay,Chandrima Paul, *Holographic helical superconductor with higher curvature corrections*,International Journal of Modern Physics A,Vol. 33, No. 1 (2018) 1850002,DOI: 10.1142/S0217751X18500021

3. Chandrima Paul, *Metallic and antiferromagnetic fixed points from gravity*, International Journal of Modern Physics A,Vol. 33, No. 16 (2018) 1850094, DOI: 10.1142/S0217751X1850094X

4. Subir Mukhopadhyay,Chandrima Paul, *Hyperscaling violating geometry with magnetic field and DC conductivity* , Nuclear Physics B 938 (2019) 571–593, DOI:10.1016/j.nuclphysb.2018.11.022

Contents

List of Figures

Chapter 1

Introduction

1.1 An Overview

Field theory have been proved to be extremely successful in describing various phenomena in Physics. Ranging from high energy theories that governs the dynamics of elementary particles to condensed matter theories involving strongly correlated systems, field theory plays dominant roles. However, its most effective machineries are essentially perturbative in nature and that creates a limitation in its applications where the interaction is strong. Though various methods such as $\frac{1}{N}$ approximation, lattice gauge theory, have been designed to tackle such problems, strong interaction remains a challenge.

In the last century a fascinating connection was established between gauge theories and theory of gravity. Actually, that such a connection exists was known for quite some time. 't Hooft [97] showed that Feynman diagrams of gauge theory using double line techniques leads to triangulation of surfaces and therefore evaluating partition function are essentially connected to sum over geometries. But it was first concretely realized through the AdS/CFT conjecture. Loosely speaking it claims that a gauge theory in d dimensions is in a sense equivalent to gravity theory in (d+1) dimensions. As it is this sounds quite unusual as gravity theories and gauge theories quite diffrenet. However, these two appearances refer to two different regimes of the coupling constants of the same theory. In particular, strongly coupled regime of gauge theory is dual to a gravity theory in weakly coupled regime and vice versa. This makes this conjecture extremely useful as it enables one to address the issues of strogly coupled regime of one theory in a weakly copuled set up of other theory. In particular, topics such as counting of degrees of freedom in a black hole solution of gravitational theory may be explored in the set up of a gauge theory. Similarly, strongly coupled phenomena of gauge theories, which were accessible to lattice computations, can be discussed in terms of a weakly coupled gravity theory. With classical gravity it can address the leading order phenomena of an $\frac{1}{N}$ expansion and it is possible to unravel the higher order contribution in a systematic manner as well.

Immediately after its advent, the correspondence have been applied to diverse arena of physics [1] . In its initial form it relates gravity theory on certain spaces called Anti de Sitter space in five dimension, denoted as AdS_5, to certain Yang-Mills theory with $SU(N)$ gauge group and N=4 supersymmetry. This duality [98] is also referred to as the holographic duality or the gauge/gravity correspondence. Though in its original formulation, the correspondence related four-dimensional Conformal Field theory (CFT) to the geometry of anti-de Sitter (AdS) space in five dimensions, soon it transpired that this duality can relate strongly correlated many body systems to the classical dynamics of gravity in one higher dimension in a much more general setting.

Theories of condensed matter turns out to be one set of natural arenas for application of this duality as frequently it involves strongly correlated systems of electrons. In particular, it was realised that this duality may be applied to explore the behaviour of superconductors and its various cousins. These kinds of systems are usually of finite temperature and finite charge density. Therefore charged black holes are natural gravity duals for them as they are characterised by finite temperatures and finite density. The dual of superconductor in its simplest version [11] considered a scalar field coupled to gravity and showed that below certain temperature, the stable configuration corresponds to black hole solution with condensation of the scalar field. In the dual theory it gets translated to a superconducting phase transition with a scalar order parameter. In this case, order parameter corresponds to spinless operator and such phases are termed as s-wave phase. Superconducting phases may corresponds to condensation of fields of non-zero spins as well, such as vectors or spin two tensor fields, which are termed as p and d-wave superconductors respectively. Examples of p-wave superconductors are UPt_3 in Sr_2RuO_4 and $(TMTSF)_2PF_6$, where the former is heavy fermion materials while the latter is organic materials. Similarly, d-wave superconductors occur in high T_c cuprate materials. In literature, black holes corresponding to all these three phases have been constructed. These black holes are usually spatially homogeneous and the superconductors are also characterised by similar homogeneity.

However, holographic superconductors may have inhomogeneous features of various kinds. In many cases, investigations have found that the homogeneus configuration suffers from instabilities giving rise to inhomogeneous ground states. Such features are present in the models of holographic QCD such as Sakai-Sugimoto model and as in certain (2+1)-dimensional brane models [99]. In the holographic applications black hole solutions are found that have instabilities leading to charge density wave or striped phases. In many of the cases, the essential mechanism responsible for such inhomogeneity is presence of Maxwell Chern-Simons term with a constant electric field. Such spatially modulated phases are widely seen in condensed matter applications.

In the present work we will begin with a certain model that gives rise to superconducting states with spatial modulation. We are motivated by the work [74], where authors have studied transition between metallic phase and an insulating phase. They considered a gravity theory coupled to two U(1) gauge fields, with the presence of Chern Simons term. This gravity theory, along with certain deformation, which can be identified as a fixed point, corresponding to metallic phase. However, an instability around this configuration triggers a flow and the system develops another fixed point solution which is related to insulating phase. Indeed, superconductors admit insulating phase characterised by antiferromagnetic properties. But incorporation of such a feature may require further generalization of this model. It was shown in [36] that antiferromagnetism of a condensed matter system is realized in terms of condensation of a scalar field in the adjoint representation of $SU(2)$, with spontaneous breakdown of symmetry $SU(2) \to U(1)$. To this end we adopt a bottom-up approach and consider a phenomenological model of gravity in five dimensions. It consists of gravity coupled to $SU(2) \times U(1)$ gauge theory with the scalar field in the adjoint representation in the presence of Chern Simons term. This model admits RN-AdS black hole as well as a helical configuration with condensation breaking $SU(2)$ to $U(1)$. As we will observe this model has a rich and complex phase structure. It admits both s-wave and p-wave phases as well as admixtures of the same. By varying the parameters we have studied the competition and coexistence of these various phases of this model in the third chapter.

One of the limitations of the analysis of the $SU(2) \times U(1)$ model stems from the fact that in this bottom up approach we have restricted ourselves to the gravity up to quadratic terms. Indeed it would have been satisfactory to incorporate the model in a string theoretic framework which will enable us to study the theory as well. Though such an incorporation has not been done, one can explore the qualitative modifications of the picture by adding suitable higher derivative terms in the gravity action. In that vein, we consider the $SU(2) \times U(1)$ model coupled to gravity theory with Gauss-Bonnet term. The reason for addition of Gauss-Bonnet term is the fact that an analogue of AdS black hole solution is admitted by it and study of behaviour of different phases with such a black hole in the background may reveal the changes in the behaviour. We have studied the effect of higher derivative terms on the boundary theory numerically and find the superconducting transition temperature will be more and more suppressed as the higher derivative terms becomes more and more strong. This result is in keeping with Mermin-Wagner theorem and is consistent with the study of role of higher derivative terms in other settings [38] .

We have also investigated similar phenomenon in the context of AdS soliton. Superconducting phase in this model is continuously connected to charged AdS black hole and below the critical temperature the superconducting phase is favoured thermodynamically. There are solitonic configuration in which charged AdS black hole may decay into, obtained by making a double Wick rotation. This undergoes a critical phase transition due to variation in chemical potential at zero temperature in the sense that there exists a critical value of chemical potential above which the scalar field condenses giving rise to a superconducting phase of AdS soliton. We have found the critical chemical potential increases with the increase of the strength of higher curvature correction. This result is also along the line of the conclusion we obtained for black hole solution. The study pertaining to higher derivative gravity appears in the fourth chapter.

Holographic theories provide natural arena for renormalisation group flow, because different slices at different values of radial coordinates may be considered as the boundary theory at different energy scales. Since in holography, couplings in the boundary theory arises as space-time fields in the gravity theory, one can consider the variation of the fields with radial distance through Callan-Symanzik equations obtained by using holographic technique. We mentioned earlier the $SU(2) \times U(1)$ model admits RN AdS black hole, whose near horizon geometry (along with certain deformation) is considered to be dual to metallic phase [74], while the condensed phase of the spatially modulated helical black hole is characterised by breaking of $SU(2)$ down to $U(1)$. Therefore it is natural to consider possible phase transitions in this model using holographic renormalization group flow between these two phases. For the present model we have obtained the gravitational Callan-Symanzik equation and β function. From the zeros of β functions we have identified two different fixed points, one of which is suggestive to be metallic nonmagnetic phase while the other turns out to be associated with spontaneous breakdown of $SU(2)$ into $U(1)$. The general phases of high temperature superconductors in Figure(1.1) suggests that such a phase may be identified with antiferromagnetic insulating configuration. It

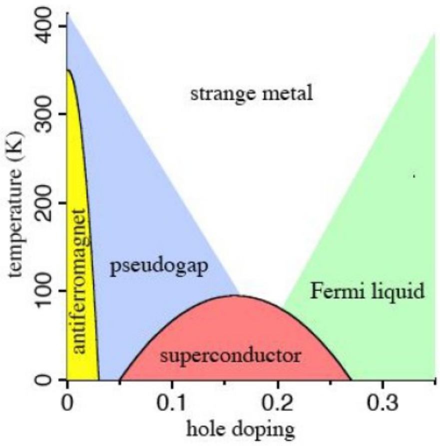

Figure 1.1: Different phases of high temperature superconductor

is known that by varying doping one can obtain such a phase in cuprate and iron based superconductors. A more explicit realisation of it would be very useful. This analysis of holographic renormalization appears in fifth chapter.

Next we turn towards transport properties. There are some materials such as heavy fermion superconductors and cuprate high T_c superconductors, which exhibit anomalous scaling of transport coefficients with temperature. That motivates us to consider hyperscaling violating theories, which may reproduce such anomalous behaviour [100]. However, such theories are substantially different from the asymptotically AdS theories that we have considered so far. In particular, we will be considering hyperscaling violating asymptotically Lifshitz theories. In such cases the boundary theories do not respect the Lorentz invariance and spatial and temporal coordinates scale in different ways. Example of such scaling can be given by the symmetry of metric under $t \to \lambda^z t \quad ; \quad \vec{x} \to \lambda \vec{x}$, where z determines the critical dimension of interactions *e.g.* $z = 1$ is relativistic invariant theory, z = 2 and 3 are characterised by onset of antiferromagnetism and ferromagnetism respectively [4].

In order to make contact with strange metallic phase of high temperature superconductor, we study the transport coefficients, such as direct conductivity, Seebeck coefficients, Hall angle etc. for such hyperscaling violating theories. One approach to study the transport properties is to compute various coefficients from the near horizon data [94], which is though quite efficient, does not accommodate different boundary conditions. In the present thesis, we have chosen a different approach [88], which admits incorporation of Dirichlet or Neumann or a mixed boundary conditions and gives rise to different transport properties for different boundary conditions. We have directly solved the equations of motion and comparing the boundary expressions we obtain thermoelectric conductivities, Hall angles etc. Though the expression of the temperature is quite involved, we have identified clean scaling regime and find, for different parameters the thermoelectric coefficients behave in different manners. These studies are accommodated in the sixth chapter of the thesis.

Since the AdS/CFT correspondence will be central machinery in these studies we need to have a clear idea about the essential structure of the correspondence. The correspondence was proposed and justified in the framework of string theory, which is a quantum theory of gravity. A concrete description of the correspondence then requires cursory review of some of the essential ingredients of string theory, gauge theory and D-brane. In the rest of this chapter we will very briefly mention the essential elements of this correpondence.

1.2 String theory and D branes

In this section we will briefly review the essetial elements of string theory and D branes following [101, 102, 117]. First we introduce the bosonic strings and superstring theory. Then we introduce the concept of D brane.

1.2.1 Bosonic strings and superstring theory

String theory is quantized theory of relativistic strings. A moving string parameterized by 1+1 dimensional world sheet described by (σ, τ) where σ is the worldsheet length and τ is worldsheet time. The string-space-time coordinates are scalar from the point of view of the world sheet, called $X^{\mu}(\sigma, \tau)$. String theory action is given by, string tension times area of the world sheet

$$S = T\int dA\,. \qquad (1.2.1)$$

In order to express this action in terms of $X^\mu(\tau,\sigma)$, we define induced metric h_{ab}

$$h_{ab} = \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}\,, \qquad (1.2.2)$$

where a,b runs over world sheet coordinate τ, σ. String action is then given by Nambu Goto action

$$\begin{aligned} S_{NG} &= \int_M d\tau d\sigma \mathcal{L}_{NG} \\ \mathcal{L}_{NG} &= \frac{1}{2\pi\alpha'}(-\mathrm{det}h_{ab})^{\frac{1}{2}}\,, \end{aligned} \qquad (1.2.3)$$

where $T = \frac{1}{2\pi\alpha'}$. We can however write it as a Polyakov action

$$S = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma \sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}\,. \qquad (1.2.4)$$

It depends on intrinsic worldsheet metric γ_{ab} and the coordinate $X^\mu(\tau,\sigma)$. Depending on boundary condition satisfied by the string coordinate $X^\mu(\tau,\sigma)$ at the spatial boundary of worldsheet, we can have two types of strings, open strings and closed strings [101].

To write the action for superstring we want to generalize Polyakov action (1.2.4) in terms of manifestly supersymmetric objects. We define

$$\Pi_a^\mu = \partial_a X^\mu + \overline{\theta}^A \Gamma^\mu \partial_a \theta^A\,, \qquad (1.2.5)$$

which are manifestly supersymmetric under global supersymmetry transformation

$$\begin{aligned} \delta\theta^A &= \epsilon^A\,;\ \delta\overline{\theta}^A = \overline{\epsilon}^A \\ \delta X^\mu &= -\overline{\epsilon}^A \Gamma^\mu \theta^A\,, \end{aligned} \qquad (1.2.6)$$

where Γ^μ is ten dimensional gamma matrices(for $SO(9,1)$) satisfies Clifford algebra $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$. So kinetic term of the action will be

$$S_{\text{kin}} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{\gamma}\gamma^{ab}\Pi_a^\mu \Pi_{b\mu} \,. \tag{1.2.7}$$

It was realized that one needs to add another term in the action, called Wess Zumino term

$$S_{wz} = \frac{1}{2\pi\alpha'} \int d^2\sigma \left[\epsilon^{ab}\partial_a X^\mu (\overline{\theta}^1 \Gamma_\mu \partial_b \theta^1 - \overline{\theta}^2 \Gamma_\mu \partial_b \theta^2) - \epsilon^{ab}(\overline{\theta}^1 \Gamma^\mu \partial_a \theta^1)(\overline{\theta}^2 \Gamma_\mu \partial_b \theta^2)\right] \,. \tag{1.2.8}$$

- The WZ term is supersymmetry invariant for $N = 2$ since we have two θ's, θ^1, θ^2, but one can write an N = 1 invariant action as well. Then the action $S_{\text{kin}} + S_{\text{wz}}$ is called Green-Schwarz action for superstring. One can define another formalism where the manifest supersymmetry is worldsheet one which is called Neveu Schwarz Ramond (NSR) action. It is written in terms of fermionic variables which are now worldsheet spinors and spacetime vector and is given by

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma [\partial_a X^\mu \partial^a X_\mu + \overline{\psi}^\mu \gamma^a \partial_a \psi^\mu] \,, \tag{1.2.9}$$

where Dirac matrices in 1+1 dimension are given by

$$\gamma^0 = \begin{vmatrix} 0 & -1 \\ +1 & 0 \end{vmatrix} = -i\sigma_2 \,, \tag{1.2.10}$$

$$\gamma^1 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = \sigma_1 \,. \tag{1.2.11}$$

The action has worldsheet supersymmetry given by

$$\begin{aligned} \delta X^\mu &= \overline{\epsilon}\psi^\mu \\ \delta\psi^\mu &= \gamma^a \partial_a X^\mu \epsilon \,, \end{aligned} \tag{1.2.12}$$

When we vary the action in order to get equation of motion, apart from the bosonic boundary term, we also get a fermionic boundary term (in open string case)

$$\psi_+\delta\psi_+ - \psi_-\delta\psi_-|_0^\pi = 0\,. \tag{1.2.13}$$

This means we have to impose the boundary condition $\psi_+ = \pm\psi_-$. We can put $\psi_+(0,\tau) = \psi_-(0,\tau)$, but then we are left with the condition

$$\psi_+^\mu(\pi,\tau) = \pm\psi_-^\mu(\pi,\tau)\,. \tag{1.2.14}$$

The condition with a + sign is called Ramond boundary condition and leads to spacetime fermionic states, and the condition with a –sign is called Neveu Schwarz (NS) boundary condition, and leads to spacetime bosonic states. Superstring theory is defined in 10 dimension. One can quantize the theory to have NS and R spectra for open strings while for closed strings we can independently put these boundary conditions for the left and right moving states, gives rise to NS NS, R R, NS R and R NS states. In 10 dimension massless particle states are classified by their behaviour under SO(8) rotation. According to SO(8) spin, we have further classification of states $NS\pm, R\pm$. A possible closed string sector thus have many possibilities, like, (R+, NS-) and so on. However it was shown in [102] that there are several constraints which removes out many possibilities. The permissible string sectors can be classified as five different superstring theories, type I, type IIA, type IIB, $E_8\times E_8$ heterotic, SO(32) heterotic. Type II theories has $N = 2$ supersymmetries, while the other three has $N = 1$ supersymmetries. In order to discuss AdS/CFT duality we will focus on supergravity limit, i.e massless spectra of the superstring theory. In $\alpha' \to 0$ limit we obtain theory of massless fields of strings. Superstring theory lives in ten dimension, include the metric $g_{\mu\nu}$, so that they describe supergravity theory, basically type IIA and IIB supergravity. For NS-NS sector, field content is $g_{\mu\nu}$, antisymmetric tensor $B_{\mu\nu}$ and a scalar ϕ. The effective action for these fields are obtained by demanding invariance under Weyl symmetry and they matches with known supergravity action.

In the case of type IIA theory RR sector contains a gauge field A_μ with field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, is called F_2 in the form language. Also there is antisymmetric tensor $A_{\mu\nu\rho}$ given by field strength $F_{\mu_1....\mu_4} = 4\partial_{[\mu_1} A_{\mu_2\mu_3\mu_4]}$, is called F_4 in the form language. The field strength of the NS NS field, $B_{\mu\nu}$ is $H_{\mu\nu\rho} = 3\partial_{[\mu} B_{\nu\rho]}$, called H_3 in the form language. The bosonic part of the supergravity action in string frame, i.e. in terms of the metric appearing in the Polyakov action for the string, is

$$S_{IIA} = \frac{1}{2\kappa_{10}^2}\int d^{10}x \left\{\sqrt{-G}\left[e^{-2\phi}\left(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}|H_3|^2\right)\right]\right\} + S_{\text{IIA,flux}} \tag{1.2.15}$$

and $S_{\text{IIA,flux}}$ is the part of the action given by RR form fields in IIA theory and their coupling to NS NS field B, given by

$$S_{\text{IIA,flux}} = \frac{1}{2\kappa_{10}^2}\int d^{10}x \left\{\sqrt{-G}\left[-\frac{1}{2}|F_2|^2 - \frac{1}{2}|\tilde{F}_4|^2\right] - \frac{1}{2}B_2 \wedge F_4 \wedge F_4\right\}, \tag{1.2.16}$$

where $|F_n|^2 = \frac{1}{n!}F_{\mu_1....\mu_n}F^{\mu_1....\mu_n}$ and

$$\tilde{F}_4 = dA_3 - A_1 \wedge F_3\,. \tag{1.2.17}$$

For type IIB supergravity, RR sector contain a scalar A_0 with field strength $F_\mu = \partial_\mu A_0$ called F_1 in form language, a two index antisymmetric tensor $A_{\mu\nu}$ with field strength $F_{\mu\nu\rho} = 3\partial_{[\mu} F_{\nu\rho]}$, called F_3 in form language, and a 4-index antisymmetric tensor field $A^+_{\mu\nu\rho\sigma}$ with modified field strength $\tilde{F}^+\mu_1....\mu_5$(also called $\tilde{F}^+_5$ in form language), which is self dual

$$\tilde{F}^+_{\mu_1...\mu_5} = \frac{1}{5!}\epsilon_{\mu_1...\mu_5}{}^{\mu_6....\mu_{10}}\tilde{F}^+_{\mu_6....\mu_{10}}\,. \tag{1.2.18}$$

Since there is the self-dual field strength consequently there is no known fully covariant form for the type IIB action. However if one imposes the self-duality as a constraint after varying the action, we have the bosonic supergravity action for type IIB string is

$$S_{IIB} = \frac{1}{2\kappa_{10}^2}\int d^{10}x \left\{\sqrt{-G}\left[e^{-2\phi}\left(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}|H_3|^2\right)\right\}\right] + S_{\text{IIB,flux}}\,, \tag{1.2.19}$$

and $S_{\text{IIB,flux}}$ is the part of the action given by RR form fields in IIB theory, given by

$$S_{\text{IIB,flux}} = \frac{1}{2\kappa_{10}^2}\int d^{10}x \left\{\sqrt{-G}\left[\frac{1}{2}|F_1|^2 - \frac{1}{2}|\tilde{F}_3|^2 - \frac{1}{2}|\tilde{F}_5|^2\right] - \frac{1}{2}A_4 \wedge H_3 \wedge F_3\right\}\,, \tag{1.2.20}$$

where we have

$$\begin{aligned} \tilde{F}_3 &= F_3 - A_o \wedge H_3 \\ \tilde{F}_5 &= F_5 - \frac{1}{2}A_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3\,. \end{aligned} \tag{1.2.21}$$

The string theory has another most important object called D brane which is an essential ingredient of AdS/CFT duality. We will discuss about it in the next sub-section.

1.2.2 D branes

We have mentioned type II theory have different form fields. In order to carry the charge of p + 1 form, one needs to introduce extended p dimensional objects. In string theory, such objects are also the ones on which open string end-points obeying Neumann boundary condition along p + 1 space time direction and Dirichlet boundary conditions in (10 –p –1) spatial directions, can attach; they are known as Dirichlet-p branes or in short Dp branes. The Dirichlet boundary conditions mean that the endpoints of open strings are fixed and the Neumann boundary conditions mean they are free and move at the speed of light. However we can consider these conditions independently for each coordinate, in particular we can choose p + 1 string coordinates to satisfy Neumann boundary conditions for p spatial dimensions of Dp brane and time and 10 –p –1 coordinates to satisfy Dirichlet boundary conditions on perpendicular direction to Dp brane. This implies that the endpoints of the open string are constrained to exist on a p + 1-dimensional wall in spacetime, which we are referring as D-p-brane. Dai,Leigh and Polchinski proved [103] that this wall where open string end point lies is dynamical, carry RR charge(as we mentiond) so that the form field as we discussed, coupled to this object. It can also fluctuate and respond to external interactions. A Dp brane has action minimizing world volume in the same way as string has an action(1.2.3) minimizing the area of its worldsheet and the coefficient of the action should be the p-brane tension, i.e. energy (mass) per unit volume, namely

$$S_p = -T_p \int d^{p+1}\xi \sqrt{-\det(h_{ab})}\,, \tag{1.2.22}$$

where h_{ab} is the induced metric on worldvolume, given by

$$h_{ab} = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b} g_{\mu\nu}(X)\,. \tag{1.2.23}$$

This is a straightforward generalization of the case of the Nambu Goto string action (1.2.3), which is expressed in terms of the induced metric on the string worldsheet.

We have seen the action(1.2.22) for a p-brane coupled to a spacetime metric. Recall NS NS sector massless spectra contains metric $g_{\mu\nu}$, antisymmetric tensor $B_{\mu\nu}$ and dilaton ϕ. In order to include the effect of complete closed string background in $\alpha' \to 0$ limit, D brane action action is given by [101, 102, 117]

$$S_p = T_p \int d^{p+1}\xi e^{-\phi} \sqrt{-\det\left(\frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b}(g_{\mu\nu} + \alpha' B_{\mu\nu})\right)}\,. \tag{1.2.24}$$

.However open string massless modes also lives on D brane gives rise to $F^{ab}F_{ab}$

term. So simplest form of bosonic D brane action is given by Dirac-Born-Infeld (DBI) action

$$S_{\mathrm{DBI}} = T_p \int d^{p+1}\xi e^{-\phi} \sqrt{-\det\left(h_{ab} + \alpha' B_{ab} + 2\pi\alpha' F_{ab}\right)} \,. \tag{1.2.25}$$

Finally we need to consider RR form fields to D brane. It was shown [117] that coupling of the form fields given by Wess Zumino action

$$S_{p,WZ} = \mu_p \int_{p+1} \left[\exp(\alpha' B_{ab} + 2\pi\alpha' F_{ab}) \wedge \sum_n A_{(n)} \right] , \tag{1.2.26}$$

where μ_p is Dp brane charge and for each given $A_{(n)}$ we keep only the term in the expansion of the exponential that completes to a p + 1-form, and of course the exponential is understood formally in terms of wedge product.

Let us consider a situation where there are N parallel D branes. So the respective open strings can have their endpoints on same D brane or different D branes. Consequently on the endpoints of open strings we can add labels $|i\rangle$, with i going from 1 to N, which are called Chan-Paton factors. Then the open strings, with one end in the N representation of U(N) and the other in the $\tilde{N}$ representation of U(N), can be considered to be in the adjoint representation [117]. Considering the matrices λ^A_{ij} in the adjoint of U(N), the open string wave functions can be expressed as

$$|k; A\rangle = \sum_{i,j=1}^{N} |k, ij\rangle \lambda^A_{ij} \,. \tag{1.2.27}$$

Therefore, when there are N parallel D-branes we expect a U(N) gauge theory. So naturally the action for scalar and U(N) gauge field are now nonabelian. At the quadratic level the action for scalar and gauge field living on D brane is given by

$$S_p = \int d^{p+1}\xi (-2) \mathrm{Tr} \left[-\frac{1}{2} D_a \vec{\phi} D^a \vec{\phi} - \frac{1}{4} F_{ab} F^{ab} + ... \right] . \tag{1.2.28}$$

In the next subsection we are going to discuss the special case of N parallel D3 branes which is the essential ingredient of AdS/CFT duality.

1.2.3 Multiple D3 branes and N=4 SYM limit

For discussion of AdS/CFT duality, the configuration of N parallel D3 branes plays an important role. Since it is known from the consideration of massless mode that the open string theory on N parallel D3 branes is $\mathcal{N} = 4$ supersymmetric gauge theory, so first we briefly review $\mathcal{N} = 4$ SYM. In 10 dimensions the minimal spinor is Majorana-Weyl, i.e out of 32 complex components of Dirac spinor, 16 real components remains. At on shell we loose half the components, left with 8 real fermionic degrees of freedom. This matches with $10 - 2 = 8$ physical degrees of freedom of vector field. Therefore the field of $\mathcal{N} = 1$ super Yang Mills in 10 dimensions are the vector A_M, $M = 0, ...9$, and the spinor ψ_π, $\pi = 1, ...16$, satisfying

$$\Gamma_{11}\psi = \psi \ ; \ \overline{\psi} = \psi^T C_{10} \,, \tag{1.2.29}$$

where Γ_{11} is the product of gamma matrices given by $\Gamma_{11} = \Gamma^0\Gamma^1\Gamma^2....\Gamma^9$. The action is given by

$$S_{10d,\mathcal{N}=1\text{SYM}} = \int d^{10}x \text{Tr}\left[-\frac{1}{4}F^{MN}F_{MN} - \frac{1}{2}\overline{\lambda}\Gamma_M D_M \,.\lambda\right] \tag{1.2.30}$$

We consider dimensional reduction of the theory from 10 to 4 dimension, as the dimension of D3 brane is. First the Γ matrices are being decomposed as

$$\Gamma_M = (\gamma^\mu \otimes \mathbb{I}, \gamma_5 \otimes \gamma_m) \,, \text{i.e } \Gamma_\mu = \gamma_\mu \otimes \mathbb{I} \ , \ \Gamma_m = \gamma_5 \otimes \gamma_m \ , \ \Gamma_{11} = \gamma_5 \otimes \gamma_7 \,. \tag{1.2.31}$$

The 10 dimensional Majorana conjugate in 4 dimensional notation is given by

$$\overline{\psi}_M = \psi^T C_4 \otimes C_6 \,, \tag{1.2.32}$$

where C_n is the charge conjugation matrix in respective dimension. Then 10-dimensional Weyl condition restricts the 10-dimensional spinors to decompose into four 4-dimensional spinors $\psi_\Pi = \psi_{\alpha i}$, $i = 1, ...4$. The 10 dimensional vector A_M decomposes 4-

dimensional vector A_μ and six scalars ϕ_m, m = 1,...6. Finally the 10 dimensional action reduces to 4 dimensional $\mathcal{N}=4$ SYM

$$
\begin{aligned}
S_{4d,\mathcal{N}=4SYM} &= (-2)\int d^4x \mathrm{Tr}[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\overline{\psi}_i\gamma^\mu D_\mu\psi^i - \frac{1}{2}D_\mu\phi_{ij}D^\mu\phi^{ij} \\
&- g\overline{\psi}^i[\phi_{ij},\psi^j] - \frac{g^2}{4}[\phi_{ij},\phi_{kl}][\phi^{ij},\phi^{kl}]] \\
&= (-2)\int d^4x \mathrm{Tr}[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\overline{\psi}_i\gamma^\mu D_\mu\psi^i - \frac{1}{2}D_\mu\phi_m D^\mu\phi^m \\
&- g\overline{\psi}^i[\phi_n,\psi^j]\overline{\gamma}^n_{[ij]} - \frac{g^2}{4}[\phi_m,\phi_n][\phi^m,\phi^n]]\,, \qquad (1.2.33)
\end{aligned}
$$

where $\phi_{[ij]} = \phi_m\overline{\gamma}^m_{[ij]}$ and $D_\mu = \partial_\mu + g[A_\mu,]$. The $\mathcal{N}=4$ SYM has SO(6) global symmetry which is known as R symmetry. The spinor ψ^i are in spinor representation of SO(6).

We consider D3 branes in type IIB superstring theory. Clearly there are 6 scalars and gauge field in the worldvolume together with fermions that fills supersymmetric multiplet. We have six scalars that have six on-shell degrees of freedom, and one 4-dimensional gauge field with two on-shell degrees of freedom, for a total of eight bosonic on-shell degrees of freedom. On the other hand, a minimal 4-dimensional fermion has two on-shell degrees of freedom. This implies for a supersymmetric theory we need to have four fermions ψ^I, $I = 1, ...4.$, all fields are in adjoint representation of U(N) gauge group (as we saw due to Chan Paton factors). So the total field content is $A^A_a, \phi^{iA}, \psi^{IA}$ that precisely matches with $\mathcal{N}=4$ SYM theory in four dimension. We can also find explicitly that the D3-brane action is invariant under four supersymmetries in four dimensions, i.e. 16 supercharges, which corresponds to half of the total supercharges of type IIB superstring theory. For D3 branes we must have the supersymmetry conditions

$$
\Gamma^0\Gamma^1....\Gamma^3\epsilon = \epsilon\,, \qquad (1.2.34)
$$

where ϵ is the supersymmetry parameter. This halves their number of components from the 32 of the type IIB theory in ten dimensions. Then we must have that the quadratic action on N D3-branes is in fact $\mathcal{N} = 4$ SYM.

1.3 Introduction to AdS/CFT correspondence

AdS/CFT correspondence relates type IIB theory on five dimensional Anti de Sitter space to Super Yang Mills in four dimension, which is a conformal field theory. In this section we will briefly review the argument that leads to conjecture of such correspondence. A first look into this correspondence shows that the isometry group of AdS_5 is $SO(4,2)$ which matches with the symmetry group of 4-dimensional conformal field theories which arises from the theory of $\mathcal{N} = 4$ SYM living on the boundary of AdS_5. On the otherhand the isometry group of S^5 is $SO(6) \simeq SU(4)$ which is same as that of R symmetry group of $\mathcal{N} = 4$ SYM.
Before reviewing the argument of the duality in details, let us introduce Anti de Sitter space and conformal group in the following two subsections. Then we will move to the correspondence.

1.3.1 The Anti De Sitter space

Anti de Sitter space plays an important role in the AdS/CFT correspondence and therefore we describe Anti de Sitter space in this subsection. The p+2 dimensional Anti De Sitter (AdS_{p+2}) space is given by the hyperboloid

$$X_0^2 + X_{p+2}^2 - \sum_{i=1}^{p+1} X_i^2 = L^2 \, , \tag{1.3.1}$$

in the flat p+3 dimensional space. The space is described by the metric

$$ds^2 = -dX_o^2 - dX_{p+2}^2 + \sum_{i=1}^{p+1} dX_i^2 \, . \tag{1.3.2}$$

In addition to the global parameterization of AdS(1.3.1,1.3.2), there is another set of coordinates $(u, t, \vec{x}, 0 < u, \vec{x} \in R^p)$, defined as

$$\begin{aligned} X^0 &= \frac{1}{2u}\left(1 + u^2(L^2 + \vec{x}^2 - t^2)\right) \ , \ X_{p+2} = Lut \\ X^i &= Lux^i \ , \ (i = 1, ..p) \\ X^{p+1} &= \frac{1}{2u}\left(1 - u^2(L^2 - \vec{x}^2 + t^2)\right) \ . \end{aligned} \tag{1.3.3}$$

These coordinates cover one half of the hyperboloid given in (1.3.1). Substituting this in(1.3.2), we obtain another form of AdS metric

$$ds^2 = L^2 \left(\frac{du^2}{u^2} + u^2(-dt^2 + d\vec{x}^2) \right) \ . \tag{1.3.4}$$

The isometry group of (d+1)-dimensional AdS space is $SO(d, 2)$.

1.3.2 Conformal Group

The correspondence relates string theory on Anti de Sitter space with conformal field theory on the boundary. Conformal field theories are the usual field theories with conformal symmetry, which attributes additional structure to them. The group associated with conformal symmetry is called conformal group. Conformal group is the group of transformation of space time which preserves the metric upto an arbitrary factor

$$g_{\mu\nu}(x) \to \Omega^2(x) g_{\mu\nu}(x) \, . \tag{1.3.5}$$

The conformal group of the Minkowski space is generated by two transformations:
1. Scale transformation

$$x^\mu \to \lambda x^\mu \, , \tag{1.3.6}$$

2.Special conformal transformation .

$$x^\mu \to \frac{x^\mu + a^\mu x^2}{! + 2x^\nu a_\nu + a^2 x^2} \, . \tag{1.3.7}$$

We denote generator of these transformation as

$M_{\mu\nu}$: for Lorentz transformation,

P_μ :for Translation,

D : for Scaling transformation,

K_μ:for Special conformal transformation.

They satisfy conformal algebra

$$[M_{\mu\nu}, P_\rho] = -i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu);\ [M_{\mu\nu}, K_\rho] = -i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu);$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = -i\eta_{\mu\rho}M_{\nu\sigma} \pm (Permutations);\ [M_{\mu\nu}, D] = 0;\ [D, K_\mu] = iK_\mu;$$
$$[D, P_\mu]\,, = -iP_\mu;\ [P_\mu, K_\nu] = 2iM_{\mu\nu} - 2i\eta_{\mu\nu}D \tag{1.3.8}$$

with all other commutators vanishing. This algebra is isomorphic to the algebra of SO(d, 2) and can be put in the standard form of SO(d, 2) algebra with generators J_{ab} by defining

$$J_{\mu\nu} = M_{\mu\nu};\ J_{\mu d} = \frac{1}{2}(K_\mu - P_\mu);\ J_{\mu(d+1)} = \frac{1}{2}(K_\mu + P_\mu);\ J_{(d+1)d} = D\,. \tag{1.3.9}$$

1.3.3 AdS CFT correspondence

Here we will review the argument connecting type IIB string theory compactified on $AdS_5 \times S_5$ to $N = 4$ Super Yang Mills theory following [98]. We will start our description from string theory side. Consider a system of N parallel D3 branes. The D3 branes are extended along a (3 + 1) dimensional plane in (9 + 1) dimensional spacetime. String theory in this background posses two types of perturbative excitations, closed string and open strings. Closed strings are excitations of 10 dimensional space whereas open strings ends on D brane and describe the excitations of D brane. If we consider the system at energy lower than string scale $1/l_s$, then only massless states will be excited. The field content of closed strings massless states is gravity supermultiplet in ten dimensions. The low energy effective Lagrangian corresponds to this gravity supermultiplet is type IIB supergravity. The open string massless states gives an $\mathcal{N} = 4$ vector supermultiplet in four dimensional world volume of D3 brane where the low energy effective Lagrangian is that of $\mathcal{N} = 4$ U(N) Super Yang Mills theory [101, 102], [104]. The complete effective action of the massless modes consists of three parts

1. Bulk action: This represents the formal action of ten dimensional supergravity and its correction with higher derivative terms

2. Brane action : This is the brane action on 3+1 dimensional D3 brane world volume and describes $\mathcal{N} = 4$ Super Yang Mills theory with higher derivative correction

3. Interaction Lagrangian: The action represents the interaction between the brane modes and bulk modes

Let us observe the complete theory in the limit, string length scale $l_s \to 0(\alpha' \to 0)$, while all the other dimensionless parameters namely string coupling constant g_s and the number of D3 branes are fixed. In this limit we also have gravitational coupling constant $\kappa \sim g_s\alpha'^2 \to 0$. So the interaction Lagrangian relating the bulk and brane vanishes. Also all the higher derivative terms in the brane action vanish (since these terms are getting α' correction), leaving just $\mathcal{N} = 4$ U(N) gauge theory in 3+1 dimensional brane worldvolume, which is known to be a conformal field theory, as we mentioned. Since the interaction Lagrangian vanishes, the supergravity theory in the bulk becomes free. We see that in this low energy limit we have two decoupled systems : free gravity in the bulk and the four dimensional gauge theory.

Here we study the same system from a different point of view. We have mentioned in the section 1.2.2 that the D branes are massive charged object which act as a source of various supergravity fields. D3 brane solution of supergravity is given by [113]

$$
\begin{aligned}
ds^2 &= g^{-\frac{1}{2}}(-dt^2 + \sum_{i=1}^{3} dx_i^2) + g^{\frac{1}{2}}(dr^2 + r^2 dS_5^2), \\
F_5 &= (1+*)dtdg^{-1}\prod_{i=1}^{3} dx_i \\
g &= 1 + \frac{L^4}{r^4}, \quad L^4 = 4\pi g_s \alpha'^2 N\,, \qquad (1.3.10)
\end{aligned}
$$

where dS_5^2 is the metric over five dimensional sphere. In this geometry, the energy E_p of an object measured by an observer at constant position r, and the energy E measured by an observer at $r \to \infty$ are related by the redshift factor

$$
E = g^{-\frac{1}{4}} E_p\,. \qquad (1.3.11)
$$

The above relation implies that the same object brought closer and closer to $r = 0$ would appear to have lower and lower energy for the observer sitting at $r \to \infty$. Let us consider the theory at low energy limit in the background described by (1.3.10). Standing at infinity an observer's observation is directed on the existence of two types of low energy excitation:

1. Massless particle propagating in the bulk with wavelength very large ,

2.One can have the kind of excitation, that one can bring closer and closer to $r = 0$.

This two types of excitations decouple from each other in the low energy limit.

The bulk massless particles get decoupled from the near horizon region (around $r = 0$) because the low energy absorption cross section goes like $\sigma = \omega^3 L^8$ where ω denotes energy. Similarly, the excitations that live very close to r = 0 find it harder and harder to climb the gravitational potential and escape to the asymptotic region. So the low energy theory described in the above geometry (1.3.10) consists of two decoupled parts, one is free bulk supergravity and the second is the near horizon region of the geometry. In the near horizon region, $r << L$, we can approximate

$g = 1 + \frac{L^4}{r^4} \sim \frac{L^4}{r^4}$, and the near horizon geometry is given by the metric

$$ds^2 = \frac{r^2}{L^2}(-dt^2 + \sum_{i=1}^{3} dx_i^2) + L^2 \frac{dr^2}{r^2} + L^2 d\Omega_5^2 \qquad (1.3.12)$$

From (1.3.4) we can identify that this is the geometry of $AdS_5 \times S^5$. We see that both from the viewpoint of a field theory of open strings living on the brane, and from the viewpoint of the supergravity description of closed string massless spectra, we have two decoupled theories in the low-energy limit. In both cases we have one of the decoupled systems is supergravity in flat space. So, it is natural to identify the second system which appears in both descriptions. Practically this leads to guess that $N = 4$ Super Yang Mills theory in 3+1 dimension is in one to one correspondence with type IIB superstring theory on $AdS_5 \times S^5$ [1] .

We will now make the conjecture more precise relating the field theory on the boundary of the AdS space and gravity in the bulk. Let us consider ϕ be the scalar field on AdS_{d+1}, coupled to gravity. Let ϕ_o be the boundary value of ϕ on the boundary of AdS_{d+1}. We will also assume that ϕ_o should be considered to couple to a conformal field O, on the boundary through the coupling $\int_{S_d} \phi_o O$. This assumption naturally appears in the AdS/CFT correspondence in the interactions of fields with branes. We would like to compute correlation function $\langle O(x_1)O(x_2)....O(x_n)\rangle$ for distinct points $x_1, x_2, ...x_n$ on the boundary of AdS_{d+1}. Let $Z_S(\phi_o)$ be the gravity partition function on AdS_{d+1} computed with the boundary condition that ϕ approaches a given function ϕ_o at infinity. In the approximation of classical gravity, one computes $Z_S(\phi_0)$ by simply extending ϕ over the boundary of AdS_{d+1} with the given boundary value where ϕ is the solution of the classical gravity equations, and then writing

$$Z_S(\phi_o) = \exp(-I_s(\phi)) \, , \qquad (1.3.13)$$

where I_s is the classical gravity action. On the otherhand one would like to define the generating functional $\langle \exp(\int_{S_d} \phi_o O)\rangle_{\rm CFT}$ (the expectation value of the exponential described here in the conformal field theory on the boundary of AdS_{d+1}). Our ansatz for the precise relation of conformal field theory on the boundary to AdS space is that

$$\langle \exp(\int_{S_d} \phi_o O) \rangle_{\mathrm{CFT}} = Z_S(\phi_o) \tag{1.3.14}$$

Hence one can compute n-point correlation function

$$\langle O(x_1)O(x_2)....O(x_n)\rangle = \frac{\partial^n}{\partial\phi_o(x_1)......\partial\phi_o(x_n)} Z_S(\phi_o)\,. \tag{1.3.15}$$

1.3.4 Gravity theory for non-zero temperature

Since we will be applying AdS/CFT correspondence in the case of strongly correlated systems at finite temperature it is relevant to explain briefly how the temperature appears on the gravity side. In order to understand the concept of temperature in the theory of gravity we need to consider a general version of AdS/CFT correspondence. The version of the correspondence that we will use asserts that conformal field theory on an n-manifold M is to be studied by summing over contributions of Einstein manifolds B of dimension n + 1. The correspondence is saying the field theory on n-manifold must have a gravity dual on B. On the field theory side one can achieve finite temperature by putting the theory on $M_n = S^1_\beta \times \mathcal{A}_{n-1}$ where Euclidean time circle S^1_β has period $\beta = 1/T$ where T is identified with temperature and $\mathcal{A}_n$ is the spatial manifold. The dual spacetime should have a natural thermal interpretation. It is a well-known fact that if we go back to the seminal works of Bekenstein and Hawking [105, 106], we can see from there that black hole spacetimes with non-degenerate event horizons exhibit features associated with thermal physics. This leads one to expect that black hole spacetimes can play a role in describing the dual of a finite temperature field theory. So we understand that in order to have a finite temperature condensed matter theory on M, we need to consider a black hole in B on the gravity side.

To summarise, in this chapter we have reviewed the essential conjecture of AdS/CFT correspondence. In the following chapters we will be considering application of this correspondence to explore various aspects of strongly correlated systems in condensed matter. In particular we will be concerned with undestanding different behaviours and properties of the superconductors using holographic techniques. Since those are finite temperature systems, on the gravity side we will be considering black hole solutions as the background.

In the next chapter we will briefly review some of the relevant aspects of holographic superconductors which will be necessary for describing the works in the following chapters.

,

Chapter 2

Introduction to holographic superconductor

In this section we will review holographic superconductor following [6, 11, 16, 20, 25, 26, 74, 76]. We have mentioned, Gauge/gravity duality has been proved to be an efficient tool to compute various aspects of strongly coupled systems in terms of weakly coupled gravity theory. Here we will discuss how this new tool comes into play for understanding high temperature superconductor. We will begin with a brief description of superconductor in condensed matter theory. Such systems can be realized on a boundary theory by using appropriate gravity model in the bulk. We review the simplest gravity model to realize the superconductor using AdS/CFT correspondence. We also describe various types of holographic superconductors namely s wave, p wave. d wave etc. within this set up.

Due to the fact that radial variable corresponds to energy scale in a gauge/gravity duality makes the latter natural arena for addressing renormalization group flow. We will mention about renormalization group flow in QFT and briefly describe how it is realized in gravity theory using AdS/CFT correspondence, which is called holographic RG flow. In order to demonstrate this, we will also consider one particular gravity model [51] and describe the realization of holographic RG flow.

Another interesting properties of superconductors are their transport properties.

Some of its phases exhibit anomalous behaviour so far transport properties are concerned. Towards the end of this chapter we will briefly review the essential techniques for studying direct conductivity of holographic superconductors.

2.1 Superconductivity

It was found that many materials, in particular metals, shows zero electrical resistivity below a certain critical temperature T_c and was termed as superconductors. Eventually it turns out their magnetic properties are quite different in the sense that they show perfect diamagnetism by expelling magnetic fields, a feature known as Meissner effect, which does not follow from zero resistivity. A phenomenological description was given by London et. al. showing decay of magnetic field inside a superconductor.

Later, Landau and Ginzberg proposed an effective theory and described superconductivity in terms of a second order phase transition. They introduced an order parameter which is a complex scalar field ϕ [108] related to density of superconducting electrons n through $n = |\phi(x)|^2$. The expression of free energy in terms of ϕ can be written as [6]

$$F = a(T - T_c)|\phi|^2 + \frac{b}{2}|\phi|^4 + ..., \tag{2.1.1}$$

where a and b are positive constants and the above series is continued to gradient and higher powers of ϕ. Because of the structure of the free energy it gives rise to spontaneous symmetry breaking as for $T > T_c$ the minimum of the free energy lies at $\phi = 0$ while for $T < T_c$, the minimum is at nonzero value of ϕ, which is given by ϕ_*,

$$|\phi_*|^2 = \frac{a}{b}(T_c - T)\,. \tag{2.1.2}$$

A more complete theory of superconductivity was given by Bardeen, Cooper and Schrieffer in 1957 and is known as BCS theory [109]. They showed that interactions of electrons with phonons can cause pairs of elections which are having opposite spins to bind and form a charged boson called a Cooper pair. Below a critical temperature T_c, there is a second order phase transition and these bosons(cooper pair) condense. Consequently DC conductivity turns out to be infinite to form a superconductor.

Subsequently various other materials were discovered to have superconductivity with higher crtical temperatures. They are cuprates (showing superconductivity along copper oxide planes), mercury-barium-copper oxide compounds. Another class of superconductors were discovered, which are based on irons instead of coppers. It is believed that for these high T_c superconductors, pair formation of electrons are responsible for superconductivity. However, the essential mechanism turns out to be due to strong coupling and differs from that of BCS. Within the regime of condensed matter not many tools are available to study such strongly coupled system. Fortunately, Gauge/Gravity provides new machinery to analyse strongly coupled field theories. In particular, this can be employed to explore transport properties of strongly coupled systems. As discussed below, we will see simple gravitational theories can reproduce superconductors and capture various features of them.

2.2 Gravitational dual of superconductor

As mentioned earlier, gravitational duals of finite temperature, finite density condensed matter systems are charged black holes. Black holes have temperature related to its surface gravity. In order to make use of the correspondence we look for black holes with asymptotically AdS spacetime. Asymptotically AdS black holes have positive specific heat like nongravitational systems.

The superconducting phases are usually characterised by condensate and this is realized on gravity side through condensation of some fields coupled to gravity. A nonzero condensate corresponds to a static nonzero field outside a black hole called black hole hair. Usually black holes do not admit hair though there are exceptions.

In superconducting materials, superconductivity arises below a critical temperature. In the dual gravitational model, this phenomena is represented through the fact that black holes does not admit hair at high temperatures. However, as temperature is lowered it shows instability towards formation of hair and thus gives to condensates. Gubser [10] and others showed that a charged black hole which is surrounded by a charged scalar field around it bears the desired property [68]. We will review the simplest holographic superconductor following [6]. The necessary action is given by gravity theory coupled to U(1) gauge field and complex scalar field,

$$
\begin{aligned}
S &= \int d^4x\sqrt{-g}[R + \frac{6}{L^2} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - |\nabla\Psi|^2 - e^2B^2|\Psi|^2 \\
&+ ieB^{\mu}(\psi\nabla_{\mu}\psi^* - \psi^*\nabla_{\mu}\psi) - m^2|\Psi|^2]\,. \qquad (2.2.1)
\end{aligned}
$$

Here charged scalar field Ψ has mass m and charge e. $F_{\mu\nu} = \partial_{\mu}B_{\nu} - \partial_{\nu}B_{\mu}$ where B_{μ} is the $U(1)$ gauge field. In this perspective the theory has been considered in the probe limit in which back reaction of the matter field has been ignored on the geometry. Consequently we consider a planar Schwarzschild anti-de Sitter black hole solution as the background geometry in four dimension. The metric and the temperature are as follows:

$$
ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2(dx^2 + dy^2)\,, \qquad (2.2.2)
$$

$$
g(r) = \frac{r^2}{L^2}(1 - \frac{r_o^3}{r^3})\,, \qquad (2.2.3)
$$

where the horizon size r_o determines Hawking temperature of the black hole

$$
T = \frac{3r_o}{4\pi L^2}\,. \qquad (2.2.4)
$$

In the probe approximation considering the metric to be given by (2.2.2,2.2.3) we solve the Maxwell and scalar equations in this background. Assuming spherically symmetric time independent ansatz for the scalar and the gauge fields,

$$\Psi = \psi(r) \qquad ; \qquad B_t = \phi(r)\,, \tag{2.2.5}$$

with $B_x = B_r = B_y = 0$ we substitute these in equations of motion. From Maxwell"s equation it turns out that the phase of ψ must be constant and without loss of generality we take ψ to be real. With this choice, the equations for the scalar field and Maxwell's equation lead to following set of coupled equations

$$\begin{aligned}\psi'' &+ \left(\frac{g'}{g} + \frac{2}{r}\right)\psi' + \frac{\phi^2}{g^2}\psi - \frac{m^2}{g}\psi = 0 \\ \phi'' &+ \frac{2}{r}\phi' - \frac{2\psi^2}{g}\phi = 0\,.\end{aligned} \tag{2.2.6}$$

As one may observe, $\frac{\phi^2}{g^2}\psi$ is coming in the opposite sign of m^2 term in (2.2.6), which provides an effective negative mass-square term to the scalar field ψ causing scalar hair to form at low temperature. We will consider the mass-square $m^2 = -\frac{2}{L^2}$, which is above BF bound, where the BF bound m^2_{BF} in four dimension is given by $m^2_{BF} = -\frac{9}{4L^2}$ [110] .

Solving the set of differential equations given in (2.2.6) require boundary conditions. For the boundary at horizon, we consider the gauge field $B_t = \phi(r)$ vanishes at horizon because of the following argument [6]. In order to describe thermal properties of the black hole, we should use the Euclidean solution. Note that the Wilson loop of $U(1)$ gauge field B_t around the Euclidean time circle is gauge invariant and finite. If B_t does not vanish at horizon, the Wilson loop remains nonzero around a vanishing circle which means that the Maxwell field is singular. So we see that $U(1)$ gauge field must vanish at the horizon in the case of a static black hole.

For the asymptotic boundary at infinity, we expect

$$\begin{aligned}\Psi &= \frac{\Psi^{(1)}}{r} + \frac{\Psi^{(2)}}{r^2} + \ldots\ldots\ldots\ldots \\ \phi &= \mu - \frac{\rho}{r} + \ldots\,.\end{aligned} \tag{2.2.7}$$

Here we will proceed with the boundary condition $\Psi^{(1)} = 0$.

Once we impose the above asymptotic boundary condition, we obtain a one parameter family of solutions, which one can find numerically [11]. We can see from (2.2.6) that ϕ is a monotonic function. It starts with the value zero at horizon and at any local maximum we have $\phi'' = \phi$. So it can neither have a positive maximum nor a negative minimum. If it starts with increasing mode away from the horizon, it continues to increase and it asymptotically approaches the constant value μ as given in (2.2.7). We also note that for $\psi(r)$, there is discrete infinite family of solutions. These solutions may not need to be monotonic.

Since we are considering the gravity theory in four dimension the dual theory is a 2+1 dimensional conformal field theory (CFT) at the temperature T (2.2.4). Properties of the dual field theory can be read off from the asymptotic behaviour of the solution of the bulk equations of motion. The asymptotic behaviour (2.2.7) of ϕ gives the chemical potential μ and charge density ρ of the field theory.

The dual theory must have an operator which is charged under the $U(1)$ and dual to ψ. As the scalar mass term close to BF bound has been chosen it has been assummed on the existence of two possible operators depending on how we quantize the theory in the bulk [111] . If we define the modes with the standard boundary condition (faster falloff) for ψ in the bulk(i.e nonzero $\Psi^{(1)}$), the dual operator has dimension two. In this context, it is assumed that there exist a nonzero $\Psi^{(1)}$ which indicates that there is a source for this operator in the CFT and the nonzero expectation value of the dual operator O_2 is indicated by nonzero $\Psi^{(2)}$

$$O_2 = \Psi^{(2)} . \tag{2.2.8}$$

Since here we want the condensate to turn on without being sourced, we set $\Psi^{(1)} = 0$. It is to be mentioned that there is a possibility of existence of alternative quantization of ψ in the bulk. In that case interchange of role of $\Psi^{(1)}$ and $\Psi^{(2)}$ can be done. It is now established that ψ is now considered to be dual to a dimension one operator. This condition can be studied considering the boundary condition remaining $\Psi^{(2)} = 0$. In that case the consideration will be made for the first case only.

In order to check superconductivity we need to know how the condensate O_2 in (2.2.8), behave with temperature T in (2.2.4). We also expect there exist a critical temperature T_C above which there will not be any condensate and this critical temperature will coincide with superconductor transition temperature. Before that discussion we need to present an important scaling symmetry. In any conformal field theory on R^n one can change the temperature by a simple rescaling. In the bulk this is given by

$$r \to ar \;\; ; \;\; (t,x,y) \to (t,x,y)/a \;\; ; \;\; r_o \to ar_o \,. \tag{2.2.9}$$

This leaves the form of the black hole metric (2.2.2) invariant with $g \to a^2 g$. Since the temperature is given by $\frac{g'(r_o)}{4\pi}$, so change of scaling factor a, changes the temperature. One can check that Maxwell field equations are invariant with rescaling (2.2.9) with $\phi \to a\phi$ and $\psi \to \psi$, i.e ψ is unchanged. In this perspective we are interested to see that how the dimensionless measure of the condensate changes as a function of a dimensionless measure of the temperature rather than discussing this trivial dependence on temperature which simply reflects the scaling dimension. It is convenient here that one can use the chemical potential μ, to fix the scale and $\sqrt{O_2}/\mu$ is studied as a function of T/μ. The curve we obtain here in Figure (2.1) is qualitatively similar to the one which is obtained from BCS theory, and observed in many materials, where we see that the condensate rises quickly as the system is cooled below the critical temperature and goes to a constant as $T \to 0$. A square root behaviour has been observed near T_c with $O_2 = 100 T_c^{\frac{3}{2}} (T_c - T)^{\frac{1}{2}}$. Also this is the expected behaviour, as predicted by Landau-Ginzburg theory. Nonzero condensate implies that the black hole develops a scalar hair. If the free energy of this hairy configuration has been computed and compared to the black hole with same charge and chemical potential but no scalar hair, it is visualized that free energy has been

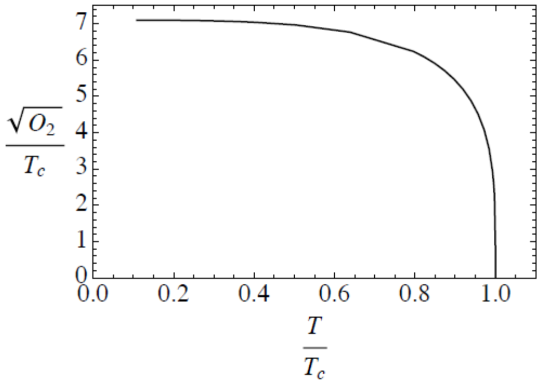

Figure 2.1: The condensate as a function of temperature

found to be always low for hairy configuration and becomes equal as $T \to T_c$ [11]. The second order phase transition has been shown by the difference of free energies scales as $(T_c - T)^2$

2.3 A.C. Conductivity

Here we give formalism for computing a.c. conductivity as a function of frequency. Let us consider a scenario, where there is an alternating vector potential of frequency ω in x direction, in the context of above model (2.2.1). The equation of motion of B_x is given by

$$B_x'' + \frac{g'}{g}B_x' + \left(\frac{\omega^2}{g^2} - \frac{2\psi^2}{g}\right) B_x = 0\,. \tag{2.3.1}$$

We are going to solve the above with ingoing boundary condition at horizon.

Asymptotically

$$B_x = B_x^{(0)} + \frac{B_x^{(1)}}{r} + \dots\,. \tag{2.3.2}$$

The gauge/gravity duality says the limit of the electric field in the bulk is the electric field on the boundary,

$$E_x = -\dot{B}_x^{(0)}\,, \tag{2.3.3}$$

and the expectation value of the induced current is

$$J_x = B_x^{(1)} \,. \tag{2.3.4}$$

From Ohm's law we get

$$\sigma(\omega) = \frac{J_x}{E_x} = \frac{J_x}{\dot{B}_x^{(0)}} = -\frac{iJ_x}{\omega B_x^{(0)}} = -\frac{iB_x^{(1)}}{\omega B_x^{(0)}} \,. \tag{2.3.5}$$

The real part of the conductivity is shown in Figure (2.2). Above the critical temperature the conductivity is constant. As we start to decrease the temperature a gap opens up at lower frequency region. There is also a delta function at the value $\omega = 0$ for all $T < T_c$. One can see it from the Drude model of a conductor. Suppose we have charge carriers with mass m, charge e, and number density n in a normal conductor. They satisfy

$$m\frac{dv}{dt} = eE - m\frac{v}{\tau} \,, \tag{2.3.6}$$

where in the above the last term is a damping term and τ is the relaxation time due to scattering. The current $J = env$. So if $E(t) = Ee^{-i\omega t}$ the conductivity is

$$\sigma(\omega) = \frac{k\tau}{1 - i\omega\tau} \,, \tag{2.3.7}$$

where $k = \frac{ne^2}{m}$. So

$$\mathrm{Re}[\sigma] = \frac{k\tau}{1+\omega^2\tau^2} \quad ; \quad \mathrm{Im}[\sigma] = \frac{k\omega\tau^2}{1+\omega^2\tau^2} \,. \tag{2.3.8}$$

A more general derivation follows from the Kramers-Kronig relations. The relation actually relate the real and imaginary parts of any causal quantity, such as the conductivity, when expressed in frequency space. One of such relation is

$$\mathrm{Im}[\sigma[\omega]] = \frac{-1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\mathrm{Re}[\sigma[\omega']]d\omega'}{\omega - \omega'} \,. \tag{2.3.9}$$

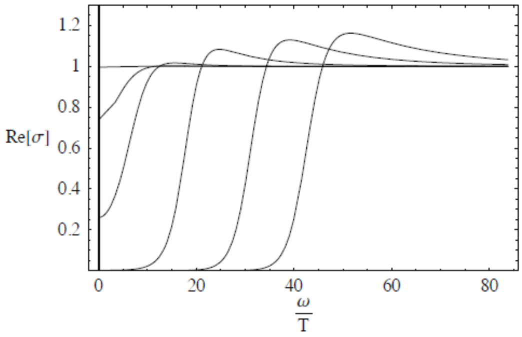

Figure 2.2: When the temperature is lowered below the level of critical temperature a gap is formed in the real part of the conductivity. The curves describe successively lower temperatures. There is also a delta function at $\omega = 0$. Figure is for the dimension two condensate

It can be visualized from the above formula that if the imaginary part of the conductivity has pole the real part of the contains a delta function. One can find there is indeed a pole in $\mathrm{Im}[\sigma]$ at $\omega = 0$ for all $T < T_c$.

2.4 s wave, p wave and d wave holographic superconductor

We have discussed in the earlier section that U(1) theory with charged scalar in the background of charged black hole solution develops instability so that scalar hair forms at low temperature. Here superconductivity is achieved by condensation of a scalar field. This is the example of s wave superconductor, which was first constructed in [11]. Similar studies about vortex-like solutions [13, 14] and anisotropic Abelian model [15] also appeared.

p wave superconductor are those materials, where superconductivity arises due to condensation of a vector field. [16–19]. Here we briefly describe an example [16] of holographic p wave superconductor. The gravity model is described by

$$S = \frac{1}{2\kappa^2} \int \mathcal{L} \,, \tag{2.4.1}$$

with

$$\mathcal{L} = R + \frac{6}{L^2} - \frac{1}{4}(F^a_{\mu\nu})^2\,, \tag{2.4.2}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc}A^b_\mu A^c_\nu\,. \tag{2.4.3}$$

Here g is SU(2) gauge field coupling constant. In an attempt to see symmetry breaking solution first we choose the metric ansatz

$$\begin{aligned} ds^2 &= e^{2a}(-hdt^2 + dx_1^2 + dx_2^2) + \frac{dr^2}{e^{2a}h}\,, \\ A &= \Phi\tau^3 dt + w(\tau^1 dx^1 + \tau^2 dx^2)\,, \end{aligned} \tag{2.4.4}$$

where $\tau^a = \frac{\sigma^a}{2i}$ where σ^a is Pauli matrices. Also a, h, Φ, ω are function of r. The boundary condition is obtained from the fact that electrostatic potential Φ must have to vanish at the horizon in order to A to be well defined at horizon. Here the condensate $w(\tau^1 dx^1 + \tau^2 dx^2)$ breaks the $U(1)$ rotation symmetry in $x^1 - x^2$ plane and also breaks $U(1)$ gauge symmetry generated by τ_3, however it preserves a combination of the two.

System of equations has a two-parameter family of black hole solutions with a non-vanishing, condensate giving vector hair.

In order to generate the d-wave condensate in the boundary theory, we have to have a massive, charged spin two field, condense in an asymptotically AdS_{d+1} geometry. Here by means of a massive spin 2 field in d+1 spacetime dimension we imply that a field that transforms locally in $(d+2)(d-1)/2$ dimensional irreducible representation of the little group SO(d) of the Lorentz group SO(1,d). The Lagrangian for such field is given by [20]

$$\mathcal{L} = \frac{1}{4}\left[-\partial_\rho\phi_{\mu\nu}\partial_\rho\phi^{\mu\nu} + 2\phi_\mu\phi^\mu - 2\phi_\mu\partial^\mu\phi + \partial_\mu\phi\partial^\mu\phi - m^2(\phi^{\mu\nu}\phi_{\mu\nu} - \phi^2)\right]\,, \tag{2.4.5}$$

where we have $\phi_\rho = \partial^\mu \phi_{\mu\rho}$ and $\phi = \phi^\rho_\rho$.

To look for a thermal state on the boundary theory in which the spin two field presumably condenses, we need to solve the equations of motion in a black hole background. Here we choose the black hole background to be

$$\begin{aligned} ds^2 &= \frac{L^2}{z^2}\left(-g(z)dt^2 + d\vec{x}^2_{d-1} + \frac{dz^2}{g(z)}\right) \\ g(z) &= 1 - \left(\frac{z}{z_h}\right)^2 . \end{aligned} \quad (2.4.6)$$

The black hole horizon is located at $z = z_h$, while the conformal boundary of the spacetime is located at z = 0. It was shown in [20], if one keeps the chemical potential μ as constant, there exist a critical temperature below which spin two field condenses.

2.5 Helical superconductors

So far we have discussed about superconducting systems which are homogeneous. However instabilities in homogeneous systems may lead to inhomogeneous configuration such as striped phases consisting of charge density waves or spin density waves, among others. Spatially modulated phases appear frequently in condensed matter systems and as mentioned in the introduction, one mechanism for obtaining such instabilities is to have a constant electric field in Maxwell Chern Simons term. Since part of our work involves such spatially modulated system, namely, one with a helical symmetry we review the essential stuff about helical system following [25, 26, 76].

Helical phases, which does not have full translational symmetry but have only a helical symmetry can be obtained by considering appropriate black hole solution in gravity theory. Helical superconductors in the context of holography was introduced in [25], [26], [76]. This was first obtained in a probe approximation by ignoring the back reaction of the matter on the geometry. They begin with a $D = 5$ action with an

$SU(2) \times U(1)$ gauge theory coupled with gravity along with a Chern-Simons term. This system admits and electrically charged AdS black brane solution. Considering linearised perturbations around such black brane in a probe approximation they have shown that there are instabilities which gives rise to condensation of vector fields leading to a p-wave superconducting phase with helical symmetry. For specific values of parameters, this analysis may also be pertinent for Roman's $N = 4$ gauged supergravity.

Subsequently, a full fledged helical black hole solution has appeared. Moreover, it was also shown that RN-AdS blck hole may decay into such configuration. The action has been chosen to be

$$S = \int d^5x\sqrt{-g}\left[R + 12 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right] - \frac{\gamma}{6}\int F \wedge F \wedge A\,. \tag{2.5.1}$$

Corresponding equations of motion are given by

$$\begin{aligned} R_{\mu\nu} &= -4g_{\mu\nu} + \frac{1}{2}\left(F_{\mu\rho}F^{\rho}_{\nu} - \frac{1}{6}F^2 g_{\mu\nu}\right) \\ d * F &+ \frac{\gamma}{2}F \wedge F = 0\,. \end{aligned} \tag{2.5.2}$$

It turns out that this system admits RN-AdS black hole as well as a helical black hole. The helical symmetry can be given as follows. For $D = 5$, considering x^1, x^2, x^3 describing the transverse directions, the killing vectors associated with the black hole are given by $\partial_{x^2}, \partial_{x^3}$ and $\partial_{x^1} - p(x^2\partial_{x^3} - x^3\partial_{x^2})$, where p is a constant (known as the pitch of the helix). They generate a helical motion consisting of a translation in the x^1 direction combined with a simultaneous rotation in (x^2, x^3) plane.

For the sake of completeness, we briefly mention the helical black hole solution following from the above action (2.5.1). The ansatz was considered in [76]

$$\begin{aligned} ds^2 &= -gf^2dt^2 + \frac{dr^2}{g} + h^2\omega_1^2 + r^2e^{2\alpha}(\omega_2 + Qdt)^2 + r^2e^{-2\alpha}\omega_3^2 \\ A &= adt + b\omega_2\,, \end{aligned} \tag{2.5.3}$$

where f , g, h, , Q, a and b are functions of the radial coordinate r only. By analysing the equations of motion we can construct the following asymptotic expansion as

$$
\begin{aligned}
g &= r^2(1-\frac{M}{r^4}+.....)\,,\\
f &= f_o(1-\frac{c_h}{r^2}+.....)\,,\\
h &= r(1+\frac{c_h}{r^4}+....)\,,\\
\alpha &= \frac{c_\alpha}{r^4}+....\,,\\
Q &= f_o(\frac{C_Q}{r^4}+....)\,,\\
a &= f_o(\mu+\frac{q}{r^2}+....)\,,\\
b &= (\frac{c_b}{r^2}+...)\,.
\end{aligned}
\tag{2.5.4}
$$

The near horizon ($r = r_+$) expansion is given by

$$
\begin{aligned}
g &= g_+(r-r_+)+.....\,,\\
f &= f_+ + ...\,,\\
h &= h_+ + ...\,,\\
\alpha &= \alpha_+ + ...\,,\\
Q &= Q_+(r-r_+)+...\,,\\
a &= a_+(r-r_+)+...\,,\\
b &= b_+ + ...\,.
\end{aligned}
\tag{2.5.5}
$$

For $\gamma > \gamma_c$ where $\gamma_c = 1.1584$ the equations of motion were solved numerically for ansatz (2.5.3) with the asymptotic boundary given by (2.5.4) and the horizon given by (2.5.5) giving rise to a black hole solution with helical symmetry.

It turns out that the stable phase of the system at high temperature is RN-AdS black hole. As the temperature decreases at some critical temperature the preferred phase corresponds to the helical black hole. An explicit thermodynamic analysis of the free energies of these two configuration confirms such transition. This helical black hole has been found to be dual to a helical current phase of the boundary system.

This helical black hole, or rather its near horizon geometry plays interesting role in another related context. In [74], authors consider a model described by action

$$\int d^5x\sqrt{-g}\left(R+12-\frac{1}{4}F_{ab}F^{ab}-\frac{1}{4}W_{ab}W^{ab}-\frac{m^2}{2}B_aB^a\right)-\frac{\kappa}{2}\int B\wedge F\wedge W\,, \tag{2.5.6}$$

where $F = dA$, $W = dB$ are field strength. One can obtain an $AdS_2 \times R^3$ along with a set of specific deformations as the solution of this system. The deformation will ensure that the solution is asymptotically AdS. This solution, as has been shown in [74] corresponds to a metallic phase.

However, as explained in [74] stability of this phase depends on the coefficients and exponents chosen for the deformation. For certain range this is stable, while for other ranges it may trigger instability. From the relations between the exponents of the deformations and the weights of the dual conformal operators it appears that the instabilities corresponds to relevant perturbations, which may lead the system to some other stable configuration. The zero temperature infrared limit of the helical black hole has been shown to be a possible stable configuration. It has also been found that it corresponds to an insulating phase. Thus such flow under relevant perturbation may be thought of as a transition from metallic to insulating phase.

2.6 Holographic renormalization group

We have seen in the previous section, there are two different configurations, which are connected through relevant perturbations and gives rise to metal-insulator transition. This provides a set up for the application of renormalisation group. Renormalization group flow in quantum field theory implies flow of the parameters of the theory with energy scale. According to AdS CFT correspondence, coupling constants etc. maps to spacetime fields and the energy scale maps to radial distance of

AdS space. From the viewpoint of gravity RG flow implies the flow of the spacetime fields along with radial distance of AdS space. In field theory the stable systems are described by zeros of β functions. β functions can be obtained from an equation involves the change of various parameters with respect to energy scale, known as Callan Symanzik equation. In the dual gravity theory one can obtain a similar equation from the quantum effective action. Like field theory case, one can also find the β function. We are interested to obtain the fixed point with simultaneous zero of all the β functions which describe the theory with certain vacuum field configuration and stable spacetime geometry.

Since in our work we have used application of holographic RG flow, we will review it with a model following [51] using Hamilton Jacobi formalism. In Hamilton Jacobi formalism, one construct the Hamiltonian with the radial direction replacing the time direction, i.e canonically conjugate momentum are defined accordingly. In a generally diffeomorphism invariant theory Hamiltonian is identically zero. Thus the hamiltonian gives sum over constraints. From these constraints only we will develop gravitational Callan Symanzik equation, which is the equation for gravitational quantum effective action and involves β functions.

We define our theory on a manifold $\mathcal{M}_{d+1}$, which has boundary Σ_d defined at constant r. Provided the radial coordinate r to the boundary is sufficiently large, Σ_d is diffeomorphic to the boundary $\partial\mathcal{M}$ at infinity. On such space, we start with the action in d+1 dimensional spacetime

$$\begin{aligned} S &= \int_{M_{d+1}} d^{d+1}x\sqrt{\gamma}\left\{V(\phi) - R_{d+1} + \frac{1}{2}L_{IJ}(\phi)\gamma^{\mu\nu}\nabla_\mu\phi^I\nabla_\nu\phi^J + \frac{1}{4}J(\phi)F^a_{\mu\nu}F^{a\mu\nu}\right\} \\ &- 2\int_{\Sigma_d} d^dx\sqrt{g}K\,. \end{aligned} \tag{2.6.1}$$

In order to keep this example general enough we have considered a system consisting of gravity, scalar field and vector fields. We have mentioned that the flow of the parameters along energy scale in the boundary theory corresponds to flow of the spacetime fields along radial direction in the bulk theory. So in order to describe the bulk dynamics in Hamiltonian language we need to decompose bulk variables in the component along radial direction and perpendicular to radial direction, since the real time is replaced by radial direction in this description. We can write the d+1 dimensional metric, under ADM decomposition using Lapse and shift function as,

$$ds^2 = (N^2+g_{\mu\nu}(x,r)N^\mu N^\nu)dr^2+g_{\mu\nu}(x,r)dr(N^\mu dx^\nu+N^\nu dx^\mu)+g_{\mu\nu}(x,r)dx^\mu dx^\nu\ , \tag{2.6.2}$$

where $g_{\mu\nu}$ is the induced metric on d dimensional hypersurface $\Sigma_d := \{X \in M_{d+1}|r = \text{constant}\}$ We wrote $\gamma = \det\gamma_{\mu\nu}$, $g = \det g_{\mu\nu}$

Using ADM decomposition (2.6.2), on each slice Σ (at constant r), we define extrinsic curvature as

$$K_{\mu\nu} = \frac{1}{2N}(\partial_r g_{\mu\nu} - \nabla_\mu N_\nu - \nabla_\nu N_\mu)\,. \tag{2.6.3}$$

The covariant derivative are defined as

$$\boldsymbol{\nabla}_\mu\phi^I = \nabla_\mu\phi^I - iA^a_\mu(T^a\phi)^I\,. \tag{2.6.4}$$

Here ∇_μ is the covariant derivative associated with Levi-Civita connection $\Gamma^\mu_{\nu\rho}$, constructed from $\gamma_{\mu\nu}$. T^a is the generator of the gauge group G. Since we want to implement ϕ as real coupling functions so we restrict the symmetry group G to a group which has real representation. Working in Hamiltonian formalism with r considered as time direction the action can be rewritten in terms of decomposition (2.6.2)

$$\begin{aligned} S &= \int d^dx dr\sqrt{g}\{\pi^{\mu\nu}\partial_r g_{\mu\nu} + \pi^I\partial_r\phi^I + \pi^{\mu a}\partial_r A^a_\mu\} \\ &+ N[\frac{1}{d-1}(\pi^\mu_\mu)^2 - \pi^2_{\mu\nu} - \frac{1}{2}L_{IJ}(\phi)\pi^I\pi^J - \frac{1}{2J(\phi)}h^{\mu\nu}\pi^a_\mu\pi^a_\nu \\ &+ V(\phi) - R_{(d)} + \frac{1}{2}L_{IJ}(\phi)h^{\mu\nu}\boldsymbol{\nabla}_\mu\phi^I\boldsymbol{\nabla}_\nu\phi^J + \frac{1}{4}J(\phi)F^{a\mu\nu}F_{a\mu\nu}] \\ &+ N^\mu[2\nabla^\nu\pi_{\mu\nu} - \pi^I\boldsymbol{\nabla}_\mu\phi^I - F^a_{\mu\nu}\pi^{a\nu}] \\ &+ A^a_r[\boldsymbol{\nabla}^a_{b\nu}\pi^{b\nu} - i\{T^a\phi\}^I\pi_I]\}\,. \end{aligned} \tag{2.6.5}$$

In Hamiltonian formalism, since the time is replaced by radial direction, so the canonically conjugate momenta to a field is defined as derivative of Lagrangian w.r.t the radial derivative of the field. Here canonical momentum conjugate to $g_{\mu\nu}$, ϕ^I, A^a_μ, respectively computed to be

$$\begin{aligned}
\pi^{\mu\nu} &= \frac{\partial\mathcal{L}_{d+1}}{\partial(\partial_r g_{\mu\nu})} = K^{\mu\nu} - g^{\mu\nu}K\,, \\
\pi^I &= \frac{\partial\mathcal{L}_{d+1}}{\partial(\partial_r \phi^I)} = \frac{1}{N}L_{IJ}(\phi)(\nabla_r\phi^J - N^\mu\nabla^\mu\phi^j)\,, \\
\pi^{a\mu} &= \frac{\partial\mathcal{L}_{d+1}}{\partial(\partial_r A^a_\mu)} = \frac{1}{N^3}J(\phi)[N^2 g^{\mu\nu}F^a_{r\nu} - N^\nu(N^2 g^{\rho\mu} + N^\rho N^\mu)F^a_{\nu\rho}]\,.
\end{aligned} \tag{2.6.6}$$

Since N, N^μ, A_r are auxiliary field in (2.6.5), their equation of motion gives first class constraint

$$\begin{aligned}
\mathcal{H} &= \frac{1}{\sqrt{g}}\frac{\delta S}{\delta N} = \frac{1}{d-1}(\pi^\mu_\mu)^2 - \pi^2_{\mu\nu} - \frac{1}{2}L_{IJ}(\phi)\pi_I\pi_J - \frac{1}{2J(\phi)}g^{\mu\nu}\pi^a_\mu\pi^a_\nu \\
&+ V(\phi) - R_d + \frac{1}{2}L_{IJ}(\phi)g^{\mu\nu}\nabla_\mu\phi^I\nabla_\nu\phi^J + \frac{1}{4}J(\phi)F^a_{\mu\nu}F^{a\mu\nu} = 0\,, \\
P_\mu &= \frac{1}{\sqrt{g}}\frac{\delta S}{\delta N^\mu} = 2\nabla^\nu\pi_{\mu\nu} - \pi_I\nabla_\mu\phi^I - F^a_{\mu\nu}\pi^{a\nu} = 0\,, \\
G^a &= \frac{1}{\sqrt{g}}\frac{\delta S}{\delta A^a_r} = \nabla^a_{b\mu}\pi^{b\mu} - i(T^a\phi)^I\pi_I = 0\,.
\end{aligned} \tag{2.6.7}$$

In the above, first two are Hamiltonian and momentum constraint respectively while third one is the Gauss's law constraint due to Gauge symmetry. Alternatively, from (2.6.5) we can write

$$\begin{aligned}
H &= \int d^dx\sqrt{g}\left[\{\pi^{\mu\nu}\partial_r g_{\mu\nu} + \pi^I\partial_r\phi^I + \pi^{\mu a}\partial_r A^a_\mu\} - \mathcal{L}\right] \\
&= \int d^dx\,[N\mathcal{H} + N^\mu P_\mu + A^a_r G^a]\,.
\end{aligned} \tag{2.6.8}$$

So from (2.6.7) we see that the Hamiltonian H is expressed as sum over constraints. From this constraint equations we are going to develop gravitational Callan Symanzik equation. Let us say we find the solution of equation of motion with the above constraints (2.6.7) under a Dirichlet boundary condition at $r = r_o$ which is given by

$$\overline{g}(x, r = r_o) = g_{\mu\nu}(x) \; ; \; \overline{A}_\mu(x, r = r_o) = A_\mu(x) \; ; \; \overline{\phi}^I(x, r = r_o) = \phi^I(x) \,. \quad (2.6.9)$$

Here the bulk fields with a bar means on-shell. Substituting the classical solution to the action, from (2.6.5) it follows that on-shell action as a functional of the boundary values

$$S[g_{\mu\nu}(x), \phi^I(x), A_\mu(x); r_o] = \int d^dx \int_{r_o}^{\infty} \sqrt{g} \left\{ \overline{\pi}^{\mu\nu} \partial_r \overline{g}_{\mu\nu} + \overline{\pi}^I \partial_r \overline{\phi}^I + \overline{\pi}^{\mu a} \partial_r \overline{A}^a_\mu \right\} \,. \quad (2.6.10)$$

Following the standard procedure in the Hamilton-Jacobi formalism, it is checked that the variation of the on-shell action(2.6.10) under the boundary values at the location of Σ_d is given by

$$\begin{aligned} \delta S[g_{\mu\nu}(x), \phi^I(x), A_\mu(x); r_o] &= \int d^dx \sqrt{g} \{ \overline{\pi}^{\mu\nu}(x, r_o) \delta g_{\mu\nu}(x) + \overline{\pi}^I(x, r_o) \delta\phi^I(x) \\ &+ \overline{\pi}^{\mu a}(x, r_o) \delta A^a_\mu(x) \} \,. \end{aligned} \quad (2.6.11)$$

We then obtain the canonically conjugate momenta for every field as the functional derivative of the on shell action w.r.t the field

$$\begin{aligned} \overline{\pi}_{\mu\nu}(x, r_o) &= -\frac{1}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}(x)} \; ; \; \overline{\pi}_I(x, r_o) = -\frac{1}{\sqrt{g}} \frac{\delta S}{\delta \phi^I(x)} \; ; \; \overline{\pi}^{a\mu}(x, r_o) = -\frac{1}{\sqrt{g}} \frac{\delta S}{\delta A^\mu_a(x)}; \\ & \frac{\delta S}{\delta r_o} = 0 \,. \end{aligned} \quad (2.6.12)$$

We now focus on Hamiltonian constraint (first equation of (2.6.7)), which reflects the invariance under radial diffeomorphism, i.e invariance of the action under local shift $r \to r + \delta r(x)$. In this constraint, we insert the expression of canonically conjugate momentum with the expressions in (2.6.12). Hamiltonian constraint then can be expressed in the following form

$$\begin{aligned}\mathcal{H} &= \{S,S\} - \mathcal{L}_d = 0 \\ \Rightarrow &\qquad \{S,S\} = \mathcal{L}_d\,, \end{aligned} \tag{2.6.13}$$

where

$$\begin{aligned}\{S,S\} &= \left(\frac{1}{\sqrt{g}}\right)^2 [-\frac{1}{d-1}\left(g_{\mu\nu}\frac{\delta S}{\delta g_{\mu\nu}}\right)^2 + \left(\frac{\delta S}{\delta g_{\mu\nu}}\right)^2 + \frac{1}{2}L^{IJ}(\phi)\frac{\delta S}{\delta \phi^I}\frac{\delta S}{\delta \phi^J} \\ &+ \frac{1}{2J(\phi)}g_{\mu\nu}\frac{\delta S}{\delta A^a_\mu}\frac{\delta S}{\delta A^a_\nu}]\,, \end{aligned} \tag{2.6.14}$$

and $\mathcal{L}_d$ is the Lagrangian reduced to d dimension and is given by

$$\mathcal{L}_d = V(\phi) - R_{(d)} + \frac{1}{2}L_{IJ}(\phi)\boldsymbol{\nabla}^\mu\phi^I\boldsymbol{\nabla}_\mu\phi^J + \frac{1}{4}J(\phi)F^a_{\mu\nu}F^{a\mu\nu}\,. \tag{2.6.15}$$

Here

$$\boldsymbol{\nabla}_\mu\phi^I = \nabla_\mu\phi^I - iA^a_\mu(T^a\phi)^I\,, \tag{2.6.16}$$

where ∇_μ denotes the covariant derivative, which is associated with Levi-Civita connection $\Gamma^\mu_{\nu\rho}$, constructed from boundary metric $g_{\mu\nu}$. The above equation (2.6.13) is identified as RG flow equation. It was shown that [51] momentum constraint and the Gausss law constraint ensures d-dimensional transverse diffeomorphism invariance and gauge invariance of the on-shell action, respectively. Following [54](see [53] for review), we consider the fact that, at the energy scale of cut off $\mu_c \sim e^{\lambda r_o}$ (where λ is characteristic constant) the action is non local. But at the energy scale $\mu << \mu_c$, a part of the action S can be represented as local action. So in that case the action S can be decomposed as local and non local part in the following way

$$\frac{1}{2\kappa^2_{d+1}}S[g(x),\phi(x),A(x)] = \frac{1}{2\kappa^2_{d+1}}S_{\text{loc}}[g(x),\phi(x),A(x)] + \Gamma[g(x),\phi(x),A(x)]\,, \tag{2.6.17}$$

where $S_{\rm loc}$ can have further derivative decomposition

$$\mathcal{L}_{\rm loc} = \sum_{w=0,2,4} [\mathcal{L}_{\rm loc}]_w \,, \tag{2.6.18}$$

where number of derivatives is denoted by w with

$$\begin{aligned} [\mathcal{L}_{\rm loc}]_0 &= W(\phi) \\ [\mathcal{L}_{\rm loc}]_2 &= -\Phi(\phi) R_d + \frac{1}{2} M_{IJ}(\phi) \nabla^\mu (\phi^I) \nabla_\mu \phi^I \,. \end{aligned} \tag{2.6.19}$$

We call the terms $W(\phi), \Phi(\phi), M_{IJ}(\phi)$ as potentials. We also define

$$S_{{\rm loc};w-d} = \int d^d x \sqrt{g} |\mathcal{L}_{\rm loc}|_w \,. \tag{2.6.20}$$

$\Gamma[g(x), \phi(x), A(x)]$ is the quantum effective action, contains higher derivative terms and the non local part. Inserting (2.6.17) into the flow equation (2.6.13), we see that the flow equation is being decomposed into couple of equations, each is identified with certain derivative number (or weight) given as supescripts as following

$$\begin{aligned} \left\{ S^{(0)}_{\rm loc}, S^{(0)}_{\rm loc} \right\} &- \mathcal{L}^{(0)}_d = 0 \,, \\ \left\{ S^{(0)}_{\rm loc}, S^{(2)}_{\rm loc} \right\} &- \frac{1}{2} \mathcal{L}^{(2)}_d = 0 \,, \\ \left\{ S^{(4)}_{\rm loc}, \Gamma \right\} + \frac{1}{2} \left\{ S^{(2)}_{\rm loc}, S^{(2)}_{\rm loc} \right\} &- \frac{1}{2} \mathcal{L}^{(4)}_d = 0 \,. \end{aligned} \tag{2.6.21}$$

Here first equation in (2.6.21) is defined with total weight 0 both side, second equation is weight 2 and so on. However we are still left with the task, that, we have to assign some weight to each component of $S_{\rm loc}$ to get certain total weight . For r.h.s of (2.6.21), we write

$$\begin{aligned} \mathcal{L}^{(0)}_d &= \sqrt{g} V(\phi) \\ \mathcal{L}^{(2)}_d &= \sqrt{g} \left\{ -R_{(d)} + \frac{1}{2} L_{IJ}(\phi) \boldsymbol{\nabla}^\mu \phi^I \boldsymbol{\nabla}_\mu \phi^J \right\} \,, \end{aligned} \tag{2.6.22}$$

and so on.

In order to completely identify the equations with different total weight w in (2.6.18 , 2.6.21), we next assign an additive number called weight to each ingredient of the action as in a table below, following [51] -

elements and weight

elements	weight
$g_{\mu\nu}(x), \phi_I(X), \Gamma[g,\phi,A]$	0
$\partial_\mu, A^a_\mu(x)$	1
$R, R_{\mu\nu}, R_{\mu\nu\rho\sigma}, \partial^2$	2
$\frac{\delta}{\delta A^a_\mu(x)}$	d-1
$\frac{\delta}{\delta g_{\mu\nu}(x)}, \frac{\delta}{\delta \phi^I(x)}$	d .

We find for weight $w=0$, flow equation is given by (first equation of (2.6.21)).

$$V(\phi) = -\frac{1}{4(d-1)}W^2(\phi) + \frac{1}{2}L^{IJ}(\phi)\partial_I W(\phi)\partial_J W(\phi)\,. \tag{2.6.23}$$

Similarly one can write the flow equation for $w=2$ and so on. For $w=d$ with d is the boundary dimension, we obtain the flow equation as

$$\begin{aligned}[\mathcal{L}_d]_d &= \frac{2\kappa^2_{d+1}W(\phi)}{2(d-1)}\frac{2}{\sqrt{g}}g_{\mu\nu}\frac{\delta\Gamma}{\delta g_{\mu\nu}} - \frac{2\kappa^2_{d+1}}{\sqrt{g}}L_{IJ}(\phi)\partial_I W(\phi)\frac{\delta\Gamma}{\delta\phi^j} \\ &- \frac{2\kappa^2_{d+1}}{gJ(\phi)}g_{\mu\nu}\frac{\delta S_{\text{loc};2-d}}{\delta A^a_\mu}\frac{\delta\Gamma}{\delta A^a_\nu} + [\{S_{\text{loc}}, S_{\text{loc}}\}]_d\,. \end{aligned} \tag{2.6.24}$$

The above (2.6.24) can be rewritten in the form of Callan Symanzik equation after some rearrangement

$$\begin{aligned} 2g_{\mu\nu}\frac{\delta\Gamma}{\delta g_{\mu\nu}} &- \frac{2(d-1)}{W}L^{IJ}\partial_I W\frac{\delta\Gamma}{\delta\phi^J} - \frac{2(d-1)}{W}\frac{1}{J}g_{\mu\nu}\frac{1}{\sqrt{g}}\frac{\delta S_{\text{loc};2-d}}{\delta A^a_\mu}\frac{\delta\Gamma}{\delta A^a_\nu} \\ &= \frac{1}{2\kappa^2_{d+1}}\frac{2(d-1)}{W}\sqrt{g}\left([\mathcal{L}_d]_d - [\{S_{\text{loc}}, S_{\text{loc}}\}]_d\right)\,. \end{aligned} \tag{2.6.25}$$

To conclude this section we mention that here we have followed Hamilton Jacobi formalism to develop RG flow equation in the context of gravity theory. Decomposing the on shell action into a local part and quantum effective action, we developed the gravitational Callan Symanzik equation. From this one can obtain β-function which in turn provides the fixed points of the theory that corresponds to the stable configurations. In chapter 5 we are going to apply the same formalism for our proposed gravity model, develop Callan Symanzik equation, identify the β function and find the fixed points and consider possible transitions.

2.7 DC Conductivity

So far we have restricted our study of different phases of holographic superconductors. Another interesting features of the condensed matter systems that we will be discussing are the transport properties. Conventional Fermi liquid theory describe systems using weakly interacting "quasi-particles" and turns out to be quite successful for all metals and semiconductors till 80's. However, afterwards new materials were discovered whose properties deviate substantially from the prediction of Fermi liquid theory. Heavy Fermion superconductors, cuprate high T_c superconductors do not have quasi-particle descriptions and with regards to transport properties cuprate materials shows linear temperature dependence of resistivity, $\rho \sim T$, $\theta_H \sim \frac{1}{T^2}$, which is different from what follows from Fermi liquid theory, $\rho \sim T^2$. This remains a challenge to understand such behaviour of strongly coupled regime using holographic methods.

It has been suggested in [100] that hyperscaling violating geometries may be the appropriate set up to look for such phenomena. One can further generalize this to hyperscaling violating Lifshitz geometries. As mentioned earlier, such geometries are characterised by two parameters z and θ, corresponding to Lifshitz scaling and the hyperscaling violation respectively. The reason for considering asymptotically Lifshitz spacetime is as follows. The canonical holographic techniques are described for asymptotically Anti de-Sitter spacetime. The symmetries of the Anti de-Sitter space implies the theories at the boundary are characterised by relativistic invariance. However, there are condensed matter systems, which instead shows anisotropic scaling symmetry along spatial and temporal directions and those cannot be addressed using a gravity theory with asymptotic AdS spacetime. Soon it was realized that the holographic techniques can be generalised to other asymptotic spacetimes as well [78–85]. In particular, for systems with anisotropic scaling, asymptotically Lifshitz spacetimes turns out to be the pertinent set up on the gravity side. There was a surge of activities, that asymptotically Lifshitz theories characterised by hyperscaling violation [86–90]. A four dimensional Einstein-Maxwell-Axion-Dilaton theory with two $U(1)$ gauge fields, usually give rise to solutions with such features. One gauge field is required to introduce Lifshitz like behaviour, while the other plays the role of electromagnetic field.

One standard method to study the transport properties is to slightly perturb the system from equillibrium, like turning on electric field, thermal gradient and evaluate the response of the system like thermal conductivity, electrical conductivity etc. If we turn on electric field E, thermal gradient ∇T, the electric current J and heat current Q are given by

$$\begin{aligned} Q &= \mathbb{K}(\nabla T) + T\alpha E\,, \\ J &= \alpha(\nabla T) + \sigma E\,, \end{aligned} \tag{2.7.1}$$

where T is the temperature, Q is the heat current, $\mathbb{K}$ is the thermal conductivity, α is thermoelectric conductivity. Using holographic techniques one can compute all these coefficients. The basic principle for such computation is as follows. Let the boundary action is perturbed by $S = S_o + \int d^dx \mathcal{J}(x)_a O_a(x)$, where $\mathcal{J}(x)_a$ is the source, which is small perturbation here and $O_a(x)$ is the dual operator on the boundary. Then Kubo's formula implies

$$\langle O_a(x)\rangle = \int d^dy G^R_{ab}(x,y)\mathcal{J}_b(y)\,, \tag{2.7.2}$$

where

$$G^R_{ab}(x,y) = \theta(x^o - y^o)\langle[O_a(x), O_b(y)]\rangle\,. \tag{2.7.3}$$

Fourier transform

$$G^R(k) = \int e^{ik(x-y)} G^R_{ab}(x,y)\,, \tag{2.7.4}$$

with

$$G^R(k) = \lim_{r\to\infty} r^{2(\triangle-d)} \frac{\Pi(r,k)}{\phi_{\rm sol}(r,k)}|_{\phi_o=0}\,. \tag{2.7.5}$$

Then using Kubo formula we obtain the relevant transport coefficients from retarded Green function. The expression of retarded Greens function can be evaluated by using the dual gravity model in holographic set up.

In general the gravity dual theories are characterised by translational invariance. However, in order to study direct conductivity we need to break the translational invariance, so that momentum conservation is not maintained. In actual condensed matter systems, in presence of an electric field, electrons undergo scattering with ionic lattices and thus dissipating the momentum giving rise to constant direct current. Therefore, in holographic set up, one needs to incorporate mechanism for such momentum dissipation. In literature, it may be realised in a varieties of way, such as choosing periodically varying chemical potential or neutral scalar field etc. One can also introduce additional neutral scalar, termed as axion and choose it to be to be linear in space coordinates to realise momentum dissipation [91–93]. In our work we have used this last mechanism.

Perhaps the simplest technique to evalaute transport coefficients is in terms of the horizon data. This method was discussed in [94, 114, 115], which makes use of the fact that certain quantities does not depend on radial coordinate and so they can be evaluated at horizon as well. The advantage lies in the fact that one need not have to consider the full solution. Though this is very efficient there are a few shortcomings of this approach. Since it is entirely focused on near horizon analysis it does not make any contact to the observables in the boundary theory. The other issue is regarding boundary condition. In general one can impose Dirichlet or Neumann boundary conditions or a mixture of them. In that sense there exists a multitude of boundary condition that may be chosen to compute the transport coefficients. However, in the near horizon approach one is restricted to a specific boundary condition (Dirichlet) only.

There is another approach, which is more suitable for identifying the boundary obserbales and incorporating general boundary conditions. In this approach, one considers linear fluctuations around the solutions and from their equation of motion one can obtain the asymptotic behaviour of physical quantites that enable one to identify the physical observable at the boundary directly. This approach also allows one to incorporate different boundary conditions. However, the expressions often diverge at boundary, which calls for introduction of appropriate counterterms. We have followed the second approach in our work.

Chapter 3

Phases of holographic helical superconductor

3.1 Introduction

Superconductors have been realized holographically in numerous models. In particular, we have already discussed s-wave superconductors, which corresponds to condensation of scalar operator [11, 12]. There are vortex like solutions [13, 14] and anisotropic Abelian model [15]. We have p-wave holographic superconductors with charged vector fields [16–18] and alternatively charged two-forms [19] and d-wave holographic superconductors which corresponds to condensation of massive spin-two fields [20]. More relevant works in these contexts have appeared in [25–34].

In this chapter we will consider a specific gravity model. As we mentioned in the introduction, we are motivated by transition between metallic phase and a non-magnetic phase studied in [74]. In view of that we consider a model with gravity coupled to $SU(2) \times U(1)$ gauge theory with the scalar field in the adjoint representation in the presence of Chern Simons term. As it transpires, this model has a quite rich and complex phase structure. We will see this model leads to holographic superconductor, with spatial modulation. Earlier it has been shown that considering only vector field in a linearised perturbation [21–26], leads to p-wave superconducting phase, with helical symmetry. The Chern Simons term can cause condensation of $SU(2)$ gauge field yielding inhomogenous superconductor. Since AdS space allows negative m^2 term, it induces instability of RN AdS black hole background towards condensation of scalar field as well. We will analyse how different superconducting phases are realised in this model and study their coexistence and competition.

It may be noted that the present model without the matter field can be incorporated in $SU(2) \times U(1)$ gauged supergravity theory. Such supergravity theory may arise as a consistent Kaluza Klein truncation of Type IIB supergravity on an S_5 [37] . In that case instability gives rise to onset of p-wave superconductor, that preserves the $U(1)$ symmetry but breaks the $SU(2)$ symmetry, reminiscent of what is seen in spiral spin density waves. However, it is not clear how to incorporate it in the string theoretic framework in presence of the matter field.

Here is a brief structure of this chapter. In section 3.1, we have discussed the model. Here we introduce probe approximation and write down the ansatz. We write down the equation of motion with the probe-metric. We also write the equations in near horizon limit and asymptotic limit. In section 3.2 we start numerical analysis. We redefine the field and write the equations of motion. We also write the expression of free energy. We obtain the plot for free energy Vs temperature. In section 3.3 we write the expression for ac conductivity. We obtain the plot for ac conductivity Vs temperature. Finally we conclude in section 3.4.

3.2 Model

As explained in the introduction, we consider a Yang-Mills gauge theory with gauge group $SU(2) \times U(1)$ coupled with gravity. In addition we also include a scalar field ϕ in adjoint of $SU(2)$. The action is represented by

$$\begin{aligned} \mathcal{L} &= (R + 12) * 1 - \frac{1}{2} G \wedge^* G - \frac{1}{2} F^a \wedge^* F^a + \frac{\gamma}{2} F^a \wedge F^a \wedge B, \\ \mathcal{L}_m &= -\frac{1}{2}[(D_\mu \phi)^{a\dagger}(D_\mu \phi)^a + m^2(\phi^{a\dagger}\phi^a)], \end{aligned} \tag{3.2.1}$$

where $G = dB$, $F^a = dA^a - \frac{1}{2}\epsilon^{abc}A^b \wedge A^c$ and $(D_\mu\phi)^a = \partial_\mu\phi^a + iB_\mu\phi^a - \epsilon^{abc}A^b_\mu\phi^c$. The scalar field couples to both the Yang-Mills gauge field in adjoint and the Abelian gauge field. m represents the mass of the scalar field. γ is the strength of Chern-Simons term consisting Abelian and non-Abelian fields. For certain value of γ the action can be obtained in string theory model in absence of scalar [37].

One can construct the equations of motion (EOM) from Lagrangian. RN-AdS black hole is one solution of EOM as given by

$$\begin{aligned} ds^2 &= -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2[(dx^1)^2 + (dx^2)^2 + (dx^3)^2], \\ g(r) &= r^2 - \frac{r_h^4}{r^2} + \frac{\mu^2}{3}(\frac{r_h^4}{r^4} - \frac{r_h^2}{r^2}), \\ B_\mu dx^\mu &= a(r)dt, \quad a(r) = \mu(1 - \frac{r_h^2}{r^2}), \end{aligned} \tag{3.2.2}$$

where B_μ is $U(1)$ gauge field, the scalar field and SU(2) gauge field are set to zero. Horizon radius of the black hole and chemical potential are r_h and μ respectively. This black hole solution has a characteristic temperature given by

$$T = \frac{6r_h^2 - \mu^2}{6\pi r_h}. \tag{3.2.3}$$

In what follows we will study instabilities associated with condensation of the scalar field and gauge field, in this background.

One can observe that the action (3.2.1) leads to equations of motion, which are quite complicated to solve. We simplify it using probe approximation [6, 10–12]. By probe approximation we mean, we will consider equations of motion of the other fields in the metric-background (3.2.2) and ignore the change in the metric. In this approximation we consider an ansatz, which may lead to a consistent solution of equations of motion. This ansatz consists of translation invariant Abelian gauge field and non abelian gauge field with a helical symmetry. A consistent ansätz with these conditions is:

$$A^1 = q(r)\,\omega_2, \quad A^2 = A^3 = 0 \quad , \quad \phi^1 = \phi^2 = 0, \quad \phi^3 = \phi(r), \qquad B = a(r)dt\,, \tag{3.2.4}$$

where $\omega_1 = \cos(kx^3)dx^1 + \sin(kx^3)dx^2$, $\omega_2 = -\sin(kx^3)dx^1 + \cos(kx^3)dx^2$, $\omega_3 = dx^3$ are one-forms, which we have introduced following [25]. In this ansätz we have both scalar and non-Abelian fields electrically charged with respect to $U(1)$ gauge field. We will analyse the equations, with this ansätz in order to find instabilities of RN-AdS black hole solution in terms of $q(r)$ and $\phi(r)$.

A more convenient form of the equations can be obtained by scaling r with horizon radius. So we write $\rho = \frac{r}{r_h}$ and replace $g(r)$ by $g_0(\rho)$ with $g(r) = r_h^2 g_0(\rho)$, where $g_0(\rho)$ is given by

$$g_0(\rho) = \rho^2 - \frac{1}{\rho^2} + (\frac{\mu^2}{3r_h^2})(\frac{1}{\rho^4} - \frac{1}{\rho^2}) \tag{3.2.5}$$

.

One can write the equations of motion in terms of these new parameters as

$$\begin{aligned}
\frac{1}{\rho^3}\partial_\rho(\rho^3 g_0(\rho)\partial_\rho\phi(\rho)) + \frac{1}{r_h^2}\frac{a(\rho)^2}{g_0(\rho)}\phi(\rho) - \frac{1}{r_h^2}\frac{q(\rho)^2}{\rho^2}\phi(\rho) - m^2\phi(\rho) = 0,\\
\partial_\rho(\rho g_0(\rho)\partial_\rho q(\rho)) - \left[\frac{k^2}{r_h^2}\frac{1}{\rho} - \gamma k\partial_\rho a(\rho) + \rho\phi(\rho)^2\right] q(\rho) = 0,\\
\partial_\rho(\rho^3\partial_\rho a(\rho)) + \frac{\gamma k}{r_h^2} q(\rho)\partial_\rho q(\rho) - \frac{\rho^3}{u_0(\rho)}\phi(\rho)^2 a(\rho) = 0.
\end{aligned} \tag{3.2.6}$$

We will now write the above equations of motion in different limits. First we consider near horizon limit which is the limit with $\rho \to 1$

- **Near Horizon limit**: In order to have near horizon limit of above equations

(3.2.6), we introduce $\rho \approx 1 + x$ and Keep the leading order terms in x. That yields,

$$\begin{aligned}
\frac{4\pi T}{r_h}(x^2\partial_x^2\phi(x) + x\partial_x\phi(x)) + \frac{1}{4\pi T r_h}a(x)^2\phi(x) - m^2 x\phi(x) = 0, \\
\frac{4\pi T}{r_h}(x\partial_x^2 q(x) + x\partial_x q(x)) - \left[\frac{k^2}{r_h^2} - \gamma k\partial_x a(x) + \phi(x)^2\right] q(x) = 0, \\
x(\partial_x^2 a(x) + 3\partial_x a(x)) + \frac{\gamma k}{r_h^2} x q(x)\partial_x q(x) - \frac{r_h}{4\pi T}\phi(x)^2 a(x) = 0.
\end{aligned} \tag{3.2.7}$$

We consider leading order solutions to be $\phi(x) \sim x^l$, $q(x) \sim x^m$ and $a(x) \sim x^n$. Substituting in equations of motion we find we need to choose $n = 0, 1$ in order to satisfy the equation of $a(x)$ at lowest order. Since at horizon, the Abelian gauge field B_0 should be zero [6, 10–12] we set $n = 1$. Other equations are satisfied for $l = m = 0$ at lowest order. However, this choice admits non-zero values at horizon for the non-Abelian gauge field, having only spatial component and the scalar.

• **Asymptotic limit**: For asymptotic limit, rewriting the equations in (3.2.6) in terms of $y = 1/\rho^2$, we get

$$\begin{aligned}
4y^3\partial_y[g_0(y)\partial_y\phi(y)] + \frac{1}{r_h^2}\frac{a(y)^2}{g_0(y)}\phi(y) - \frac{1}{g_0^2} y q(y)^2\phi(y) - m^2\phi(y) = 0, \\
4y^2\partial_y[y g_0(y)\partial_y q(y)] - \left[\frac{k^2}{r_h^2} y + 2\gamma k y^2\partial_y a(y) + \phi(y)^2\right] q(y) = 0, \\
4y\partial_y^2 a(y) - \frac{\gamma k}{r_h^2} y\partial_y q(y)^2 - \frac{1}{y^2 g_0(y)}\phi(y)^2 a(y) = 0, \\
\text{where} \quad g_0(y) = \frac{1}{y} - y + (\mu^2/3r_h^2)(y^2 - y).
\end{aligned} \tag{3.2.8}$$

Asymptotic limit of the above equation can be obtained from leading order terms in a Taylor expansion around $y = 0$. Substituting $\phi(y) \sim y^\alpha$, $q(y) \sim y^\beta$ and $a(y) \sim y^\delta$ as leading order solution, at lowest order we obtain from ϕ equation

$$\alpha = \triangle_\pm = 1 \pm \sqrt{1 + m^2/4}. \tag{3.2.9}$$

In order to find scalar field condensation, we need to choose negative m^2 term and we set $m^2 = -3$ for convenience, which gives an asymptotic solution for ϕ, $\phi(y) = \phi_0 y^{1/2} + \phi_1 y^{3/2}$. Both the solutions corresponds to normalisable modes. Following standard quantisation we set the boundary condition to be $\phi_0 = 0$. Similarly, allowed values of β are 0 or 1 implying $q(y) \sim q_0 + (q_1/r_h^2)y$. here normalisable mode corresponds to $q_0 = 0$. $a(y)$ is chosen to be $a(y) \sim (\mu - \frac{a_1 y}{r_h^2})$. In the next section we will look for solutions of these equations numerically.

3.3 Numerical Analysis

Since we want to get rid of fractional power of ϕ we substitute new variable $\chi(y) = \frac{\phi(y)}{\sqrt{y}}$. Withat modification, equations for $\chi(y)$, $q(y)$ and $a(y)$ are given by,

$$
\begin{aligned}
&4y^3 g_0(y)\partial_y^2\chi(y) + 4y^2\partial_y(y g_0(y))\partial_y\chi(y)\\
&+\left[(2y^2\partial_y g_0(y) - y g_0(y)) + \frac{1}{r_h^2}\frac{a(y)^2}{g_0(y)} - \frac{1}{r_h^2} y q(y)^2 + 3\right]\chi(y) = 0,\\
&4y\partial_y^2 a(y) - \frac{\gamma k}{r_h^2} y\partial_y(q(y)^2) - \frac{1}{y g_0(y)}\chi(y)^2 a(y) = 0,\\
&4y^2 g_0(y)\partial_y^2 q(y) + 4y\partial_y(y g_0(y))\partial_y q(y) - [\frac{k^2}{r_h^2} + 2\gamma k y\partial_y a(y) + \chi(y)^2]q(y) = 0,
\end{aligned}
\tag{3.3.1}
$$

where $g_0(y)$ is given in (3.2.8). The boundary conditions at $y = 0$ changed to:

$$
\chi(y) \approx \frac{\chi_1}{r_h^2}y, \quad q(y) \approx \frac{q_1}{r_h^2}y, \quad a(y) \approx (\mu - a_1\frac{y}{r_h^2}). \tag{3.3.2}
$$

However, it should not admit normalisable mode for q at $k = 0$ for the reason given below. For $k = 0$ second equation of (3.2.6) takes the form

$$
q'' + \left(\frac{u_0'}{u_0} + \frac{1}{\rho}\right)q' - \frac{\phi^2}{u_0}q = 0. \tag{3.3.3}
$$

For $\rho \to 1$ we get $q'(1) = \frac{r_h}{4\pi T}\phi(1)^2 q(1)$, and so q and q' has same sign at horizon. Then q is positive and increasing (or negative and decreasing) at horizon. A normalisable mode implies it vanishes at $\rho \to \infty$ and so there is some intermediate value $\rho = \rho_t$, where q turned around leading to a maximum (or minimum). Then equation

(3.3.3) is not satisfied at $\rho = \rho_t$ as there q and q'' have opposite signs with $q' = 0$.

We have used mathematica software to numerically solve these equations (3.3.1), which are written in probe approximation, with boundary condition (3.3.2). We have used the following values for the parameters: $\mu = a_1 = 0.1$ and $\gamma = 1$ and solve for the values of χ and q for which $a(x)$ vanishes at horizon.

In order to determine thermodynamic stability of various phases and compare them we have evaluated free energy density. Free energy density is given by Euclidean action on shell. The expression for free energy we obtained to be

$$F = \frac{TS_E}{V} = -\mu a_1 - \frac{1}{2}\int_{r_h}^{\infty}[-\gamma k q q' a - r\phi^2 a + \frac{r^3}{u(r)}\phi^2 a^2]dr. \qquad (3.3.4)$$

In fig. (3.1) we have plotted free energy of s, p and $s + p$ wave phases against temperature. We have restricted the range near the critical temperature. From the plot one can see that at a given temperature, s+p-wave has the highest energy, and free energy of s-wave is the lowest. Free energy of p-wave phase comes in between. Therefore thermodynamically preferred phase is s-wave always compared to other phases. One can also observe from the plot that phase transitions are second order. Similarly values of the condensates are plotted vs. temperature near critical temperature as given in Figure (3.2).

We have also studied the pitch dependence of free energy vs. temperature plot for p-wave phase in Figure (3.3). The plots are made for three different values of k. We see for $k = 1.5, 3.5$ and 5, as given in left subfigure free energy increases with k, while it decreases for $k = 0.05$ and 0.1 (middle subfigure). This implies that free energy will be minimum at a critical value of k near $k = 0.1$ indicating thermodynamically preferred phase. Once again we obseerve the phase transitions are of second order in nature.

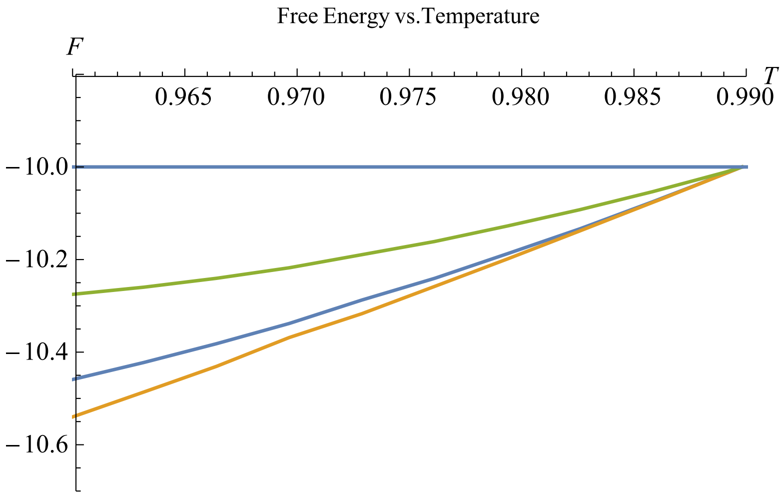

Figure 3.1: Plot of free energy vs temperature for normal, s+p, p and s-wave phases (from top to bottom along F-axis) on left for $k = 1.5$

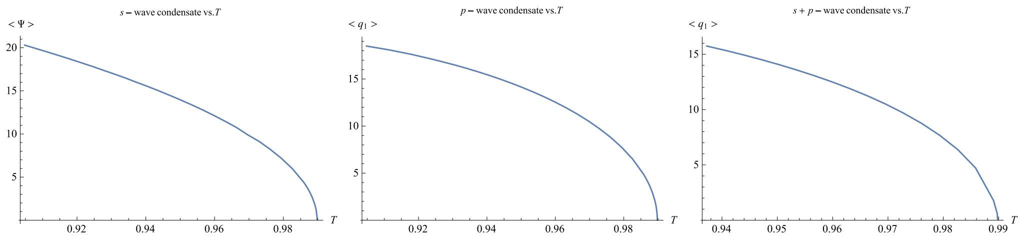

Figure 3.2: Plot of s-wave (left) and p-wave (middle) and s+p-wave (on right) condenstates vs. temperature

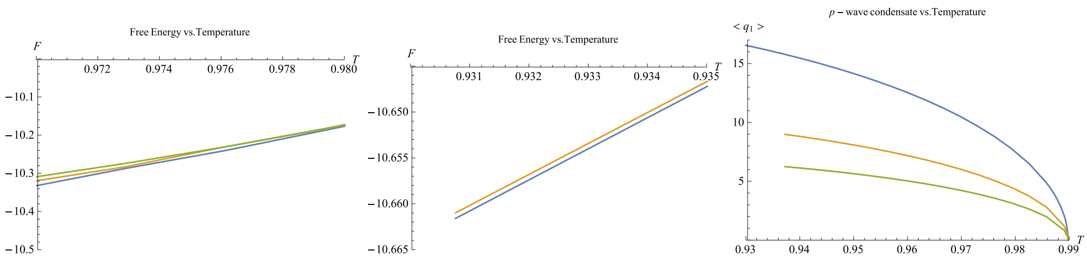

Figure 3.3: Plot of p-wave free Energy (left) and condensate (on right) for $k = 1.5, 3.5, 5$. Plots from top to bottom correspond to decreasing k on left and increasing k on right. The middle figure is plot of free energy for $k = 0.1$ (bottom) and $k = 0.05$ (top)

3.4 Conductivity

We discuss the optical conductivity of the system as the function of frequency in this section. One has to develop some numerical techniques [11, 38, 39] to achieve the same. Since the Abelian gauge field B_μ couples to a current J_μ, so to study conductivity we introduce fluctuation of B_μ as a linear perturbation and write the equations of motion for it. We are working with inhomogeneous background which carries a background momentum k with background gauge field $B_\mu^{(0)}$ which is the solution of our equations of motion. We consider a small perturbation around this background as $B_\mu = B_\mu^{(0)} + b_\mu$. We follow [22] to write

$$b_\mu dx^\mu = b_L(t,r)dx^3 + b_T(t,r)\omega_2 \tag{3.4.1}$$

It follows that b_L and b_T staisfy the following equations

$$\begin{aligned} b_L''(r) + \left(\frac{g'(r)}{g(r)} + \frac{1}{r}\right) b_L' + \left(\frac{\omega^2}{g(r)^2} - \frac{\phi(r)^2}{g(r)}\right) b_L(r) = 0, \\ b_T''(r) + \left(\frac{g'(r)}{g(r)} + \frac{1}{r}\right) b_T' + \left(\frac{\omega^2}{g(r)^2} - \frac{\phi(r)^2}{u(r)} - \frac{k^2}{r^2 g(r)}\right) b_T(r) = 0, \end{aligned} \tag{3.4.2}$$

where we consider the facts that the components of U(1) gauge fields are given by time dependent function as $b_{L,T}(t,r) = b_{L,T}(r)e^{-i\omega t}$. We also impose ingoing wave boundary condition at the horizon

$$b_{L,T}(r) \sim g(r)^{-i\frac{\omega}{4r_0}}, \tag{3.4.3}$$

to insist the fact that there will be no outgoing radiation at horizon. $b_L(r)$ and $b_T(r)$ behave asymptotically as [22, 39]

$$\begin{aligned} b_L(r) = b_L^{(0)} + \frac{b_L^{(2)}}{r^2} + \frac{b_L^{(0)}}{2}\omega^2 \frac{Log(\Lambda r)}{r^2} + ..., \\ b_T(r) = b_T^{(0)} + \frac{b_T^{(2)}}{r^2} + \frac{b_T^{(0)}}{2}(\omega^2 - k^2)\frac{Log(\Lambda r)}{r^2} + ..., \end{aligned} \tag{3.4.4}$$

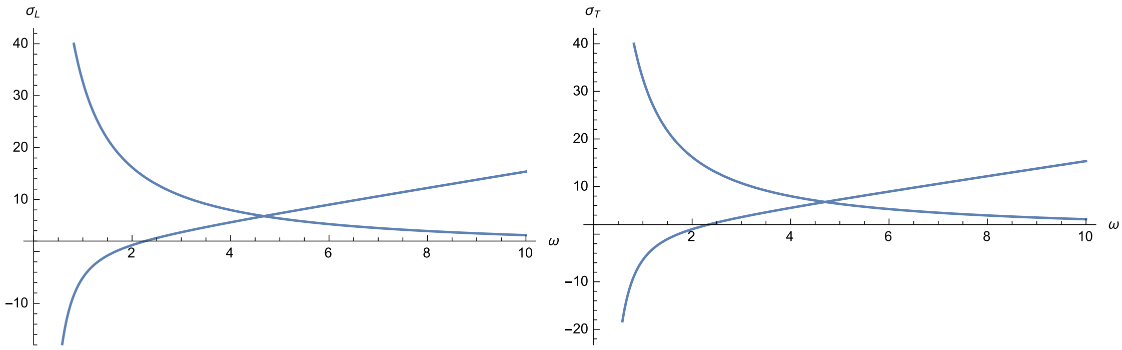

Figure 3.4: Plot of longitudinal (left) and transverse (right) conductivities vs. frequency

where $b_L^{(0)}$, $b_L^{(2)}$, $b_T^{(0)}$, $b_T^{(2)}$ and Λ are integration constants.

Since unlike the case of [22], $b_L(r)$ and $b_T(r)$ decouple in our case, we consider the diagonal components of retarded Green function, which are as follows [39]

$$\begin{aligned} G^R_{LL} &= 2\frac{b_L^{(2)}}{b_L^{(0)}} + \omega^2 \left(log(\Lambda r) - \frac{1}{2} \right), \\ G^R_{TT} &= 2\frac{b_T^{(2)}}{b_T^{(0)}} + (\omega^2 - k^2) \left(log(\Lambda r) - \frac{1}{2} \right). \end{aligned} \tag{3.4.5}$$

We remove the logarithmic divergence with boundary counterterm in the action. After we cancel the logarithmic term, we obtain the expression for the conductivity, which are given as following [22],

$$\sigma_L(\omega) = \frac{2b_L^{(2)}}{i\omega b_L^{(0)}} + \frac{i\omega}{2}, \quad \sigma_T(\omega) = \frac{2b_T^{(2)}}{i\omega b_T^{(0)}} + \frac{i}{2\omega}(\omega^2 - k^2) \tag{3.4.6}$$

We have computed both the real and imaginary parts of longitudinal and transverse conductivities. We have plotted them against frequency $\tilde{\omega} = \omega/r_h$ in Figure (3.4) . As ω approaches to zero, the imaginary parts shows a divergence. This signifies a $\delta(\omega)$ behaviour for the respective real parts. This behaviour can be observed in the case of superconductors.

3.5 Conclusion

We have shown in the probe approximation the model with gravity coupled U(1) and SU(2) gauge field with scalar field in the adjoint representation, along with a Chern-Simons term admits s and p wave phases as well as their coexistence. We have found the expression for thermodynamic free energy and studied the free energy plot with temperature for different phaes as well as for p-wave phases with different values of pitch. We find s-wave phase is thermodynamically more stable than p wave phase which is even more stable than s+p wave phase. So we may conclude the fact that s wave phase is a consistent ground state which breaks global $SU(2)$ to $U(1)$. We found therefore, in thermodynamically preferred phase only the scalar field condenses. For p-wave phases, we consider different values of pitch and we studied thermodynamic free energy. We found free energy becomes minimum for a critical value of pitch. From the plots of free energies it is being observed that the phase transitions are of second order in nature. However, probe approximation fails whenever the condensates have large values.

This work may be extended in several directions. We have considered only one chemical potential associated with the $U(1)$ field and for a certain value of CS coupling. A natural generalisation of our work, therefore, would be to turn on electric field for $SU(2)$ fields. Phase structure of 3 parameter space with two chemical potentials and CS coupling would give a richer phase structure. Secondly, due to complexity of the equations of motion we restricted ourselves to probe approximation. A complementary study with back reaction of the gravity taken into consideration will provide a complete picture. One can introduce neutral scalar field and make the coupling of the interaction dependent on them along the line of [35]. By tuning the field dependent coupling it may provide dual to the antiferromagnetic phase [36].

Another issue is the instability that we find in this analysis is valid for classical gravity theory. In general, however, one may consider higher curvature corrections, inclusion of which may provide information regarding modification of the boundary theory as we move away from large ‘tHooft limit. One can ask whether the instabilities found here will survive such corrections. As discussed elsewhere [38] it could be that higher curvature corrections may remove these instabilities or may further modify the phase structure. This is the issue that we will discuss in the next chapter.

Chapter 4

Higher curvature corrections

4.1 Introduction

In the last chapter we have discussed our proposed model i.e gravity coupled to U(1) and SU(2) gauge field with the scalar field in adjoint. We have analysed its phase structures and compare their stabilities thermodynamically. However, the analysis was restricted to classical gravity theory, which corresponds, in gauge/gravity parlance, large N and large 't Hooft limit. As we have mentioned at the end, it would be interesting to see how the instability and the phase structure in general gets modified once we move away from this limit by including higher derivative terms in the gravity theory.

One reason that why such higher derivative term would lead to a difference is as follows [38]. Holographic superconductors are associated with spontaneous breaking of a $U(1)$ symmetry. Such superconducting phases have been found in the holographic set up in almost any dimension. However, spontaneous breaking of continuous symmetry is forbidden in $(2+1)$-dimensions according to Mermin-Wagner theorem due to large fluctuations in lower dimensions. But $(2+1)$-dimension admits holographic superconductors, at least within the regime of classical gravity. It could then well be, that at the large N limit, fluctutations are suppressed in the case of holographic superconductors. As one deviates from the large N limit by incorporating higher curvature corrections, the fluctuations will be dominant and the condensation of the field giving rise to superconducting phase will be suppressed.

Ofcourse, in order to check for such a phenomenon one need to consider four dimensional gravity theory. However, four dimensional gravity theories with higher derivative, such as Gauss-Bonnet and Lovelock gravity uses specific combinations of curvature tensor and becomes non-dynamical in four dimensions. Other theories that uses powers of Ricci scalar leads to identical black hole solutions as in Einstein case. More general higher derivative gravity theories involve ghost degrees of fredom. nevertheless, one can expect to have similar phenomenon in five dimensions as well.

In view of that we consider the proposed model along with Gauss-Bonnet terms. In presence of Gauss-Bonnet term, the gravity theory admits a black hole solution, known as Gauss-Bonnet black hole. Within a probe approximation we consider the $SU(2) \times U(1)$ gauge theory along with the charged scalar and study the modification of the behaviour of the condensate and the associated phase structure.

In addition, we will also study a chemical potential driven transition in this theory in presence of an AdS soliton background. Holographic superconductor transition occurs between two phases as a change of temperature. As shown in [41], a similar transition occurs at zero temperature due to variation in chemical potential when the holographic superconductor is placed in an AdS soliton background. By compactifying one of the space direction of AdS black hole, one can obtain AdS soliton configuration, which decays into AdS black hole through Hawking-Page transition at a critical temperature. In the AdS soliton background, the phase where charged field condenses, termed as AdS soliton superconductor has a much larger gap and may be considered as an anlogue of an insulator [41]. Since the AdS soliton is obtained by compactifying one space direction of the asymptotic AdS spacetime, the dual theory is a Scherk-Schwarz compactification of a four dimensional gauge theory. This analysis demonstrates the effects of higher curvature in a 2+1 dimensional theory.

This chapter is organised as follows. In section 4.2 we introduce the model which is gravity coupled to SU(2) gauge field and U(1) gauge field with scalar field in adjoint with higher curvature correction. Here we introduce AdS soliton background and AdS black hole background, write down the analytic expression of free energy in both the background. We also write down the boundary condition on the field in both the backgrounds with both the boundaries. In section 4.3 we discuss the numerical results obtained for normal phase and superconducting phase for both AdS black hole and AdS soliton. Here we present free energy Vs temperature and condensate Vs temperature plot. We conclude in section 4.4.

4.2 Model

Let us recall that our model consists of gravity coupled to $SU(2) \times U(1)$ Yang Mills fields with a scalar field in the adjoint representation in the presence of Chern-Simons coupling between Abelian and non-Abelian fields. It has been explained in [21] that the Chern-Simons term plays an important role in condensation of the vector field. To include the higher curvature terms, we consider Einstein-Gauss-Bonnet action for the gravitational part. The modified Lagrangian takes the following form

$$\begin{aligned}
\mathcal{L} &= (R+12)*1 + \frac{\alpha}{2}(R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2) - \frac{1}{2}G \wedge^* G \\
&- \frac{1}{2}F^a \wedge^* F^a + \frac{\gamma}{2}F^a \wedge F^a \wedge B \\
\mathcal{L}_m &= -\frac{1}{2}[(D_\mu\phi^\dagger)^a(D^\mu\phi)^a + m^2\phi^{a\dagger}\phi^a], \qquad (4.2.1)
\end{aligned}$$

where $G = dB$ and $F^a = dA^a - \epsilon^{abc}A^b \wedge A^c$ represents field strength of Abelian and non-Abelian gauge fields respectively. Covariant derivatives are given by $(D_\mu\phi)^a = \partial_\mu\phi^a + iB_\mu\phi^a - \epsilon^{abc}A^b_\mu\phi^c$. Riemann curvature tensor, Ricci tensor and Ricci scalar are given by $R_{\mu\nu\rho\sigma}$, $R_{\mu\nu}$ and R respectively.

•AdS Black Hole:

In this case, we will be considering a neutral black hole solution which is the solution of equations of motion from the action [43, 44] given by,

$$ds^2 = -g(\rho)dt^2 + \frac{d\rho^2}{g(\rho)} + \rho^2 \left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2)\right], \tag{4.2.2}$$

where

$$g(\rho) = \frac{\rho^2}{2\alpha}\left[1 - \sqrt{1 - \frac{4\alpha}{L^2}\left(1 - \frac{ML^2}{\rho^4}\right)}\right]. \tag{4.2.3}$$

M is related to the AdS mass of the black hole and the black hole horizon is expressed as $g(\rho_0) = 0$ is

$$\rho_o = (ML^2)^{1/4}. \tag{4.2.4}$$

Here L represents the radius of the asymptotics AdS space. The strength of the higher curvature terms is given by the parameter α. For $\alpha = 0$, $g(\rho)$ becomes $\frac{\rho^2}{L^2} - \frac{M}{\rho^2}$. In order to avoid naked singularity we restrict it to $\alpha \leq 1/4$. The temperature of the black hole is

$$T = \frac{u'(\rho)}{4\pi}|_{\rho=\rho_o} = \frac{\rho_o}{\pi L^2} = \frac{(M)^{\frac{1}{4}}}{\pi(L)^{\frac{3}{2}}}. \tag{4.2.5}$$

•AdS Soliton:

We are also going to consider AdS soliton solution. One can obtain AdS soliton [45, 46] solution in two steps. First we make a double wick rotation of the AdS black hole metric (4.2.2). Then we compactify one of the space direction in a circle, such that the radius of it shrinks down to zero at some finite value of r. The dual theory corresponds to a $2+1$-dimensional theory, which is a Scherk-Schwarz compactification of a four-dimensional comformal gauge theory. In presence of higher curvature [47] correction, the metric of the AdS soliton becomes

$$ds^2 = \left[\frac{d\rho^2}{g(\rho)} + g(\rho)d\chi^2 + \rho^2(-dt^2 + dx_1^2 + dx_2^2)\right], \tag{4.2.6}$$

where the expression of $g(\rho)$ is given in (4.2.3). We have made χ periodic to obtain a smooth space and the associated temperature can be considered to be zero.

Here we consider the matter fields of our model in the above AdS black hole and AdS soliton backgrounds. We consider following ansatz for the gauge and scalar fields, to simplify the equations of motion. In particular, we choose ansatz for the Yang-Mills fields to be spatially inhomogeneous depending on pitch k.

$$A^1 = q(\rho)\omega_2, \quad A^2 = A^3 = 0 \quad , \quad \phi^1 = \phi^2 = 0, \quad \phi^3 = \phi(\rho), \quad B = a(\rho)dt, \tag{4.2.7}$$

where, once again, we are using one forms $\omega_1 = \cos(kx^3)dx^1 + \sin(kx^3)dx^2$, $\omega_2 = -\sin(kx^3)dx^1 + \cos(kx^3)dx^2$, $\omega_3 = dx^3$ introduced earlier. We substitute the above ansatz in the equations of motion for each of the backgrounds.
Here first we consider AdS black hole background.

•AdS Black Hole background:

For AdS black hole substituting black hole metric as given in (4.2.2, 4.2.3), the ansatz (4.2.7) gives the following equations of motion:

$$\begin{aligned}
&\partial_\rho(\rho^3 g(\rho)\partial_\rho\phi(\rho)) + \frac{\rho^3 a(\rho)^2}{g(\rho)}\phi(\rho) - \rho q(\rho)^2\phi(\rho) - m^2\rho^3\phi(\rho) = 0,\\
&\partial_\rho(\rho g(\rho)\partial_\rho q(\rho)) - (\frac{k^2}{\rho} - \gamma k\partial_\rho a(\rho) + \rho\phi(\rho)^2)q(\rho) = 0,\\
&\partial_\rho(\rho^3\partial_\rho a(\rho)) + \gamma k q(\rho)\partial_\rho q(\rho) - \frac{\rho^3}{g(\rho)}\phi(\rho)^2 a(\rho) = 0.
\end{aligned} \tag{4.2.8}$$

From the Maxwell's equation we can see that phase of the complex scalar is constant. We set the phase of this complex scalar field to be zero.

Since we aim to solve these equations of motions (4.2.8) we need to choose right boundary conditions at near horizon and asymptotic infinity. In order to choose near horizon boundary condition, at near horizon limit we consider the leading behaviour of the fields, from the equations of motion. Expanding the equations (4.2.8) around horizon and ensuring that they will not diverge there, we obtain the following consistency conditions at the near horizon limit:

$$\begin{aligned}
&4\rho\partial_\rho\phi(\rho) - \left(\frac{q(\rho)^2}{\rho^2} + m^2\right)\phi(\rho) = 0, \\
&4\rho^2 q'(\rho) - \left(\frac{k^2}{\rho} - \gamma k(\partial_\rho a(\rho)) + \rho\phi(\rho)^2\right) q(\rho) = 0, \\
&a(\rho) = 0. \qquad (4.2.9)
\end{aligned}$$

The behaviour of the fields at asymptotic infinity, i.e $\rho \to \infty$ given by

$$\begin{aligned}
a(\rho) &= \mu - \frac{\rho_1}{\rho^2}, \\
\phi(\rho) &= \frac{C_-}{\rho^{\lambda_-}} + \frac{C_+}{\rho^{\lambda_+}}, \\
q(\rho) &= q_1 - \frac{q_2}{\rho^2}. \qquad (4.2.10)
\end{aligned}$$

μ and ρ represent chemical potential and charge density respectively of the system. Charge density will be kept fixed, which provides a scale. The exponents in (4.2.10), $\lambda_\pm$ are

$$\lambda_\pm = 2 \pm \sqrt{4 + \frac{2m^2\alpha}{1 - \sqrt{1 - 4\alpha}}}, \qquad (4.2.11)$$

where we will choose $m^2 = -3$ for the matter fields. Hence it can be seen from (4.2.11) both the modes in the scalar fields are normalisable modes. As per standard quantisation procedure we set $C_- = 0$. We will also set $q_1 = 0$. Then C_+ and q_2 will give v.e.v of dual operator or the respective condensates.

For the purpose of thermodynamic analysis we consider free energy associated with different phases. Free energy density of the system is given by integrating the Euclidean action on shell. For a general configuration consistent with the ansatz (4.2.7) expression for free energy comes out to be

$$\begin{aligned}
F &= \frac{TS_E}{V} \\
&= -\mu\rho_1 - \frac{1}{2}\int_{\rho_o}^{\infty} d\rho \left[-\gamma k q(\rho)(\partial_\rho q(\rho))a(\rho) - \rho\phi(\rho)^2 q(\rho)^2 + \frac{\rho^3}{u(\rho)}\phi(\rho)^2 a(\rho)^2\right] \\
&. \qquad (4.2.12)
\end{aligned}$$

•AdS Soliton background:

Here we will study the phases of the system in AdS soliton background (4.2.6). We consider equations of motion for the matter fields (4.2.13) in this background, consistent with the ansatz (4.2.7). Equations of motion are as follows

$$
\begin{aligned}
&\frac{1}{\rho^3}\partial_\rho(\rho^3 g(\rho)\partial_\rho\phi(\rho)) + \frac{a(\rho)^2}{\rho^2}\phi(\rho) - \frac{q(\rho)^2}{\rho^2}\phi(\rho) - m^2\phi(\rho) = 0,\\
&\frac{1}{\rho^3}\partial_\rho(\rho u(\rho)\partial_\rho q(\rho)) - \frac{k^2}{\rho^2 u(\rho)}q(\rho) + \gamma k\partial_\rho a(\rho)q(\rho) - q(\rho)\frac{\phi(\rho)^2}{\rho^2} = 0,\\
&\frac{1}{\rho^3}\partial_\rho(\rho g(\rho)\partial_\rho a(\rho)) + \gamma k q(\rho)\partial_\rho q(\rho) - \frac{a(\rho)}{\rho^2}\phi(\rho)^2 = 0.
\end{aligned}
\qquad (4.2.13)
$$

Here we need to find boundary conditions for the fields, both at the tip of the soliton and at asymptotic infinitiy. At the tip, $\rho = \rho_0$, the matter fields behave as follows,

$$
\begin{aligned}
\phi(\rho) &= a_1 + b_1 ln(\rho - \rho_o) + c_1(\rho - \rho_o),\\
q(\rho) &= a_2 + b_2 ln(\rho - \rho_o) + c_2(\rho - \rho_o),\\
a(\rho) &= a_3 + b_3 ln(\rho - \rho_o) + c_3(\rho - \rho_o).
\end{aligned}
\qquad (4.2.14)
$$

The logarithmic terms represents divergences of the respective fields at the tip. In order to avoid the divergence we need to impose consistency condition, that the coefficients of the logarithmic terms in the expansion of the fields (4.2.14) should be equal to zero.

At the asymptotic infinity, $r \to \infty$, the behaviour of the fields are quite similar to that in the case of AdS black hole background (4.2.10) . The leading order expressions are given as before

$$\begin{aligned} a(\rho) &= \mu - \frac{\rho_1}{\rho^2}, \\ \phi(\rho) &= \frac{C_-}{\rho^{\lambda_-}} + \frac{C_+}{\rho^{\lambda_+}}, \\ q(\rho) &= q_1 - \frac{q_2}{\rho^2}. \end{aligned} \tag{4.2.15}$$

$\lambda_\pm$ are given by (4.2.11) and μ and ρ represent the chemical potential and charge density of the boundary theory. We choose $C_- = 0$. The v.e.v of the dual operator associated with the scalar field is represented by C_+. For vector fields, we set $q_1 = 0$, then q_2 will represent the value of the condensate.

Free energy density associated with this AdS soliton background is obtained from the Euclidean action on shell and comes out to be

$$F = \frac{TS_E}{V} = -\mu\rho_1 - \frac{1}{2}\int_{\rho_o}^{\infty} d\rho \left[-\gamma k q(\rho)(\partial_\rho q(\rho))a(\rho) - \rho\phi(\rho)^2 q(\rho)^2 + r\phi(\rho)^2 a(\rho)^2\right]. \tag{4.2.16}$$

In the next section, we will numerically solve the equations (4.2.8) and (4.2.13) subject to the consistency conditions (4.2.9) and (4.2.14) respectively at horizon and asymptotic condition given by (4.2.10), (4.2.15). In the case of AdS black hole, we will use the solutions to study the behaviour of condensates and free energies with variation of temperature. Similar study will be discussed for AdS soliton background with variation of chemical potential.

4.3 Numerical Analysis

In this section we discuss the numerical results obtained for free energy and condensate in normal phase and superconducting phase for both AdS black hole and AdS soliton background. For AdS black hole background we solve the equations (4.2.8) with asymptotic boundary conditions as given in the following,

$$\begin{aligned} a(\rho) &\simeq \mu - \frac{\rho_1}{\rho^2}, \\ \phi(\rho) &\simeq \frac{C_+}{\rho^{\lambda_+}}, \\ q(\rho) &\simeq \frac{q_2}{\rho^2}. \end{aligned} \tag{4.3.1}$$

In order to ensure that fields do not diverge at horizon, the solutions need to satisfy the consistency conditions given in (4.2.9) .

We keep the charge density ρ_1 fixed and find solutions, which gives rise to non-zero condensate C_+, while $C_- = 0$. In order to study the change of scenario with the variation of the coefficient of Gauss-Bonnet term, α, it has been chosen to be $\alpha = 0.1$ and $\alpha = 0.05$. We have given the plot of condensates of scalar field vs. temperature (i.e C_+ vs temperature) in Fig.4.1. As one can observe that the condensation gap becomes larger as coefficient of Gauss-Bonnet term increases. Since the scalar has a negative m^2 term within BF bound, as well as a potential well that forms near horizon [38], one can presume that condensation will happen. We observe critical temperature decreases with increase in α. Such a decrease in transition temperature with α is also confirmed from the plot of the free energy vs. temperature in Fig.4.2.

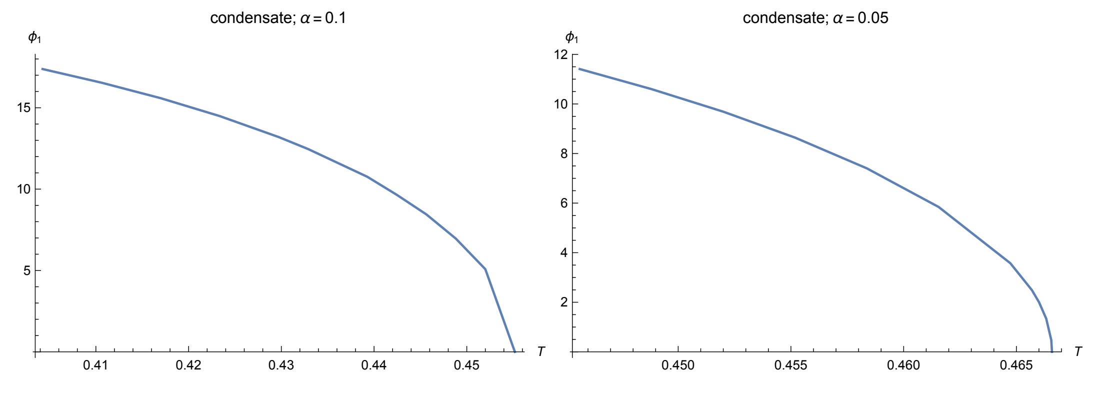

Figure 4.1: Condensate vs. temperature plot for s wave $\alpha = 0.1$ (left) and $\alpha = 0.05$ (right)

It also admits p-wave phase where the non abelian gauge field gets v.e.v leading to symmetry breaking. This will gives rise to a configuration with helical symmetry due to the choice of our ansatz(4.2.7). One may observe from the plots of condensate

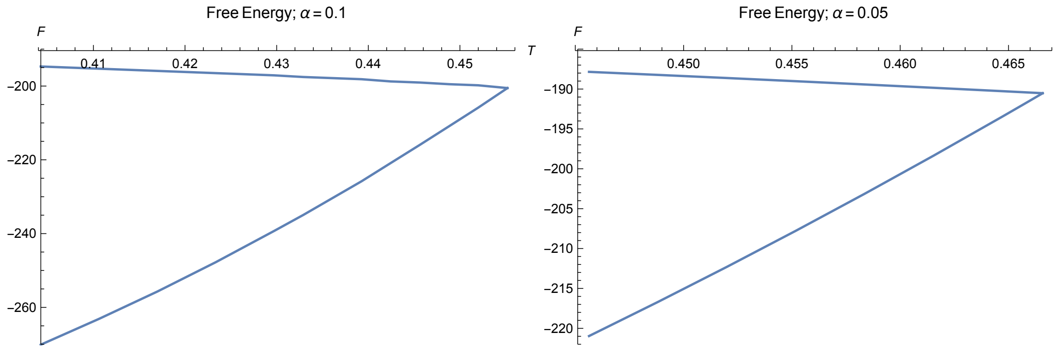

Figure 4.2: Free Energy vs. temperature plot for s wave superconducting phase(below) and normal phase(above) for $\alpha = 0.1$(left) and $\alpha = 0.05$ (right)

vs. temperature for $\alpha = 0.1$ and $\alpha = 0.05$ for p-wave phase, given in Fig.4.3 that the gap increases as the coefficient of the higher curvature term gets larger. Similarly, from the free energy vs. temperature plot, given in Fig.4.4 we see the transition temperature decreases with increase of the strength of the higher curvature correction, as in the s wave case. It is evident from free energy vs. temperature plot for both cases of s-wave and p-wave that the nature of the phase trasitions are of second order.

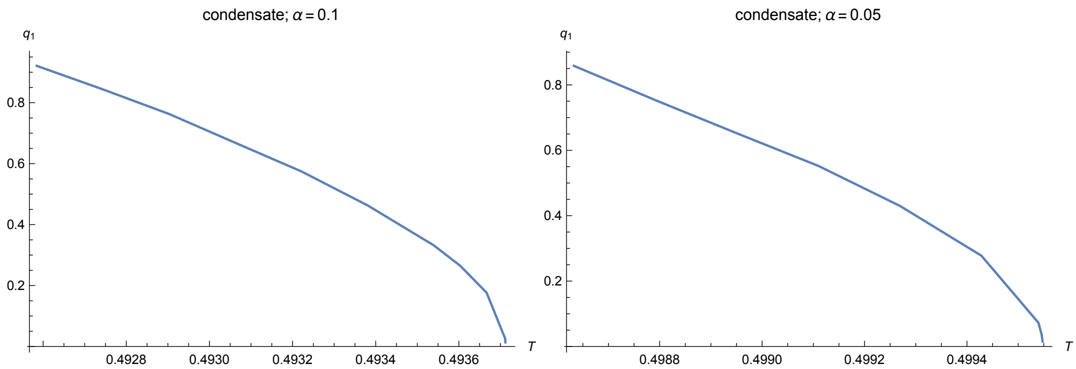

Figure 4.3: Condensate vs temperature plot for p wave for $\alpha = 0.1$ (left) and $\alpha = 0.05$ (right)

Next we make similar study in AdS soliton background. For this background we will consider the s-wave only. We numerically solve (4.2.13), subject to the asymptotic boundary condition (4.3.1) and consistency condition (4.2.14) at the tip so that coefficient of logarithmic term is zero. The plot of condensate of the scalar field and free energy against chemical potential μ are given in Fig.4.5 and Fig.4.6 for $\alpha = 0.05$ and $\alpha = 0.1$. Charge density (ρ) has been plotted vs chemical potential

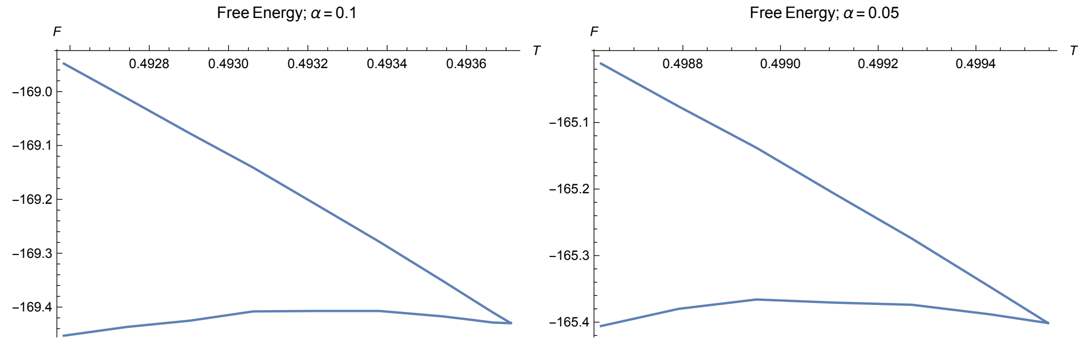

Figure 4.4: Free Energy vs temperature plot for p wave superconducting phase(below) and normal phase(above) $\alpha = 0.1$(left) and $\alpha = 0.05$ (right)

in Fig.4.7 for the two values of α. One may note that the critical value of μ at which the transition takes place, increases with the increase of the strength of the higher curvature terms. The plot of free energy vs. μ for the two values of α also exhibits the same as given in Fig.4.6.

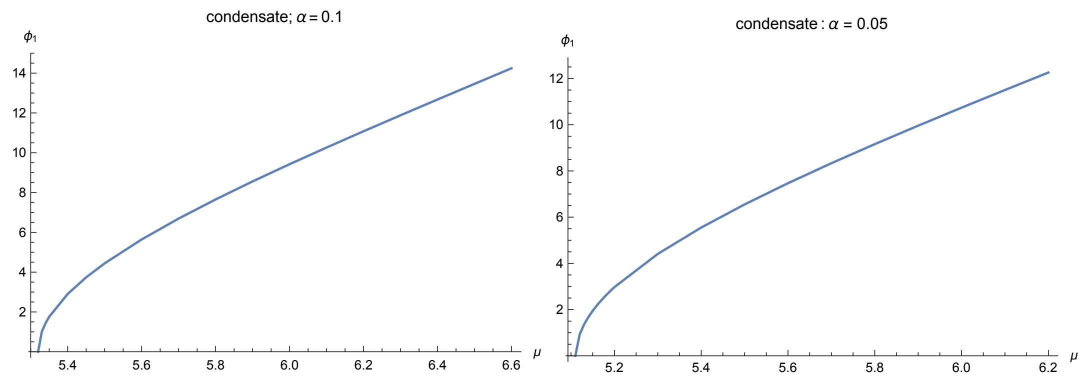

Figure 4.5: condensate vs μ plot for s wave phase for $\alpha = 0.1$(left) and $\alpha = 0.05$ (right) in AdS soliton background

4.4 Conclusion

In this chapter we have studied phases of the model introduced earlier in presence of higher curvature corrections. It admits a Gauss-Bonnet black hole solution and in this background we find that below a critical temperature, one gets superconducting phase. It is now established that the magnitude of decrease in critical temperature is closely linked with the increase of the strength of the higher curvature term. Therefore, one can conclude that as the strength of the higher curvature terms increases

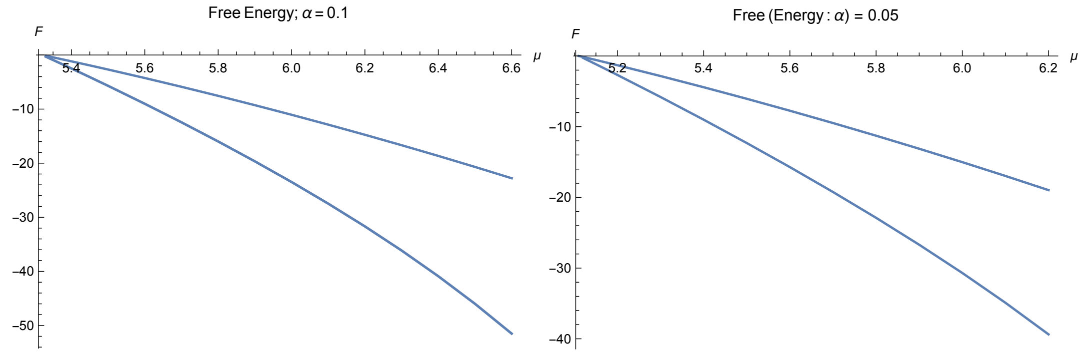

Figure 4.6: Free Energy vs. μ plot for normal phase(above) and s wave superconducting phase(below) for $\alpha = 0.1$ (left) and $\alpha = 0.05$ (right) in AdS soliton background

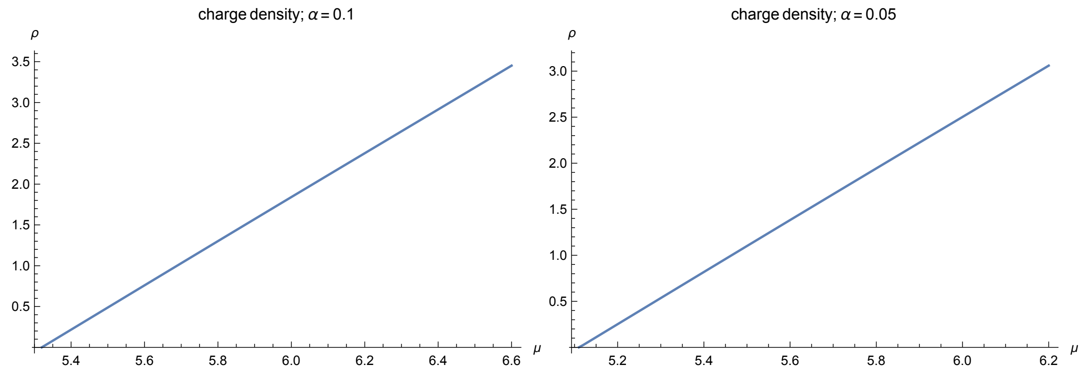

Figure 4.7: ρ vs. μ plot for s-wave phase for $\alpha = 0.1$ (left) and $\alpha = 0.05$ (right) in AdS soliton background

the transition gets more and more suppressed and it will be more difficult for the condensation to take place. This is in agreement with the result obtained in the case of simple s-wave black hole [38]. It may be interesting to find the value of α for which condensations cease to occur.

In the next part of the analysis, we considered AdS soliton background, at zero temperature for which the dual theory is (2+1)-dimensional. As we vary the chemical potential, we find phase transition occurs at some critical value $\mu = \mu_c$. We study the impact of the higher curvature corrections on this transition and find as we increase the strength of the higher curvature terms the critical value of the chemical potential increases. Therefore, with increase of higher curvature corrections, the AdS soliton phase will persist and one requires higher value of chemical potential in order to have condensation of the charged fields. In other words, it shows suppression of condensation due to existence of higher curvature terms in (2+1)-dimension.

As discussed in the introduction Mermin-Wagner theorem forbids spontaneous breaking of a continuous symmetry in (2+1)-dimension due to the fact that massless fluctuations destroy the long range order. Holographic superconductors appear in (2+1) dimension in classical gravity. Since the latter corresponds to the large N limit of the field theory, the suppression of massless fluctuations can be attributed to the large N limit [12,48]. Therefore, as we deviate from the large N limit we should find that condensation is getting suppressed [38]. As we have seen, with increase of the strength of higher curvature term, the critical temparature, at which superconducting transition takes place decreases or critical value of chemical potential increases. Both the phenomena show suppression of condensation, which is in accordance with this possibility. For the AdS soliton, the dual theory is (2+1) dimensional. For the black hole, though the dual theory is (3+1) dimensional one may expect a similar phenomenon in (2+1) dimension as well.

It will be interesting to find out the behaviour of spatially separated correlators of the massless fluctuations within an AdS/CFT framework [12]. One can also try to look for an analogue of Berezinski-Kosterlitz-Thouless transition in this system. Once again, we have ignored the back reaction on the metric and considering the back reaction one may obtain a more detailed understanding of the effects due to the higher derivative corrections. Another set of features to capture the effect of moving away from large N limit may be transport properties. In particular, one can study thermoelectric properties of various phases at higher curvature along the line of [94].

Chapter 5

Holographic Renormalisation Group Flow

5.1 Introduction

As we have discussed in the previous chapters that gauge/gravity correspondence has been applied successfully in the realm of condensed matter physics [4–9] (and references therein), and especially in the context of holographic superconductors ([11] - [20]). High temperature superconductors turns out to have rich phase structures and it is interesting to understand transitions among the various phases from a holographic viewpoint. Renormalisation Group (RG) flow has been used in the study of such transitions which are called quantum phase transition. In particular, transition between metallic and insulating phase was analysed in [74], where these two phases were described as fixed points of RG flow. Similar works in the study of metallic and insulaing fixed point also appeared in [75].

We are interested in studying the phases in the present model using techniques of holographic RG flow used in [53, 54]. According to holographic duality a stable phase corresponds to certain stable field configuration and appear as the fixed point of holographic RG flow. An instability of the phase corresponds to flow between one fixed point to the other, i.e one geometric configuration and vacuum field configuration to the other. We have obtained the potential and by minimising it we have identified the fixed points as metallic and antiferromagnetic phases.

Our work is partially motivated from the fact that there is significant development in the study of holographic renormalization group ([49]- [67]), in recent studies. In particular, we have used the concept from the work by Verlinde et.al., who proposed some formalism [53–55,67] which has wide applications in AdS/CFT correspondence. According to it an on shell action can be written as the sum of local and non local part. Buik equations of motion can be written using Hamilton Jacobi formalism so that they look formally similar to RG equations. Many other works also have appeared in this context [65, 66] (and references therein). Subsequently, studies of RG in the context of dilaton theory appeared in literature [49, 50, 52, 57]. It was developed futher to include the gauge fields in the dilaton theory action [51].

Here is the brief organization of the present chapter. In section 5.2 we introduce our model, develop the RG formalism and construct Callan Symanzik equation. Here also we derive the potential equation. In section 5.3 we describe our first fixed point, which is AdS fixed point. We show that this fixed point is related to metallic phase. In section 5.4 we describe the nontrivial fixed point and show that this fixed point is related to antiferromagnetic phase. In section 5.5 we derive certain stability condition of metallic phase. We conclude in section 5.6. The complete equation of motions of the field with full back reacted metric is written in Appendix A. In Appendix A we have also expressed the expression of the fields and some of the metric components in terms of the potentials as we are going to introduce in the following sections.

5.2 Derivation of Callan Symmanzic equation

In chapter 2.6 we have demonstrated holographic RG flow for a gravity model, described in [51]. In this section we are going to apply the techniques of holographic RG flow to our proposed model [40]. Recall that it is gravity coupled to

$SU(2) \times U(1)$ gauge theory, with the scalar in the adjoint in the presence of Chern Simons term and for the present purpose we have added Gibbons Hawking term. The action is given by,

$$
\begin{aligned}
S &= \int_M d\rho d^4x\sqrt{-g}[R - \Lambda - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}W_{\mu\nu}W^{\mu\nu} - \frac{1}{2}(D_\mu\phi^a)^\dagger(D_\mu\phi^a) \\
&- \frac{m^2}{2}\phi^{a\dagger}\phi^a] - \frac{\kappa}{2}\int B\wedge F\wedge W + \int_{\partial M}\sqrt{-\gamma}d^4x 2K\,, \qquad (5.2.1)
\end{aligned}
$$

where $F = dA, W^a = dB^a - \epsilon^{abc}B^b \wedge B^c$ and $(D_\mu\phi)^a = \partial_\mu\phi^a + iA_\mu\phi^a - \epsilon^{abc}B^b_\mu\phi^c$, with A and B are U(1) and SU(2) gauge fields respectively, ϕ is the scalar in SU(2)$\times$ U(1) adjoint representation. Also Λ is cosmological constant and here in the present case it is given by $\Lambda = -12$. Note that the above action (5.2.1) is the same action, we have introduced in (3.2.1) except we have added the last term, which is Gibbons-Hawking term where γ^{ij} is boundary metric. Gibbons Hawking boundary term is ensuring that this action admits a Hamiltonian description. We choose the metric ansatz

$$
ds^2 = -g(\rho)dt^2 + \frac{d\rho^2}{U(\rho)} + ds^2_{\text{transverse}}\,, \qquad (5.2.2)
$$

where

$$
\begin{aligned}
ds_{\text{transverse}^2} &= e^{2x_1(\rho)}\omega_1^2 + e^{2x_2(\rho)}\omega_2^2 + e^{2x_3(\rho)}\omega_3^2 \\
&= e^{2x_1}dy_1^2 + \left(e^{2x_2}\text{Cos}^2 ky_1 + e^{2x_3}\text{Sin}^2 ky_1\right)dy_2^2 \\
&+ \left(e^{2x_3}\text{Cos}^2 ky_1 + e^{2x_2}\text{Sin}^2 ky_1\right)dy_3^2 \\
&+ 2(e^{2x_3} - e^{2x_2})\text{Cos}(ky_1)\text{Sin}(ky_1)dy_2dy_3\,, \qquad (5.2.3)
\end{aligned}
$$

where once again we use one forms $\omega_1 = dy^1$, $\omega_2 = \text{Cos}(ky^1)dy^2 - \text{Sin}(ky^1)dy^3$, $\omega_3 = \text{Sin}(ky^1)dy^2 + \text{Cos}(ky^1)dy^3$. Here $x_i(\rho), U(\rho)$ are radial functions and k is a parameter representing the pitch. Asymptotically AdS geometry implies at $\rho \to \infty$ we have $x_i(\rho) \to ln\rho$ and $U(\rho) = \rho^2$. Note that this (5.2.3) is the same as the metric ansatz used in [74], except in our case g_{tt} is different from their case. Also recall that in section 3.2 we have written probe approximation of this metric in (3.2.2). Here we are considering full backreacted metric. For our purpose we impose simplifying ansatz for the fields, which is consistent with the equations of motion, as we considered in (3.2.4):

$$B^1 = w(\rho)\omega_2, \quad B^2 = B^3 = 0 \quad , \quad \phi^1 = \phi^2 = 0, \quad \phi^3 = \phi(\rho), \quad A = a(\rho)dt\,. \tag{5.2.4}$$

We have discussed that in the holographic RG formalism in the dual field theory which describes the flow of parameters with energy scale, in the bulk theory that describes the flow of the fields with radial direction ρ. Note that our aim is to describe the bulk dynamics in a Hamiltonian language where the Hamiltonian is now the generator of translation in radial direction instead of time direction. So we start with decomposing bulk variables along and perpendicular to radial direction as in the standard ADM treatment of gravity. First we start with the bulk metric which is written with lapse and shift variables in the form [52], [53]

$$ds^2 = (N^2 + N^i N^i)d\rho^2 + 2N^i d\rho dy^i + \gamma_{ij} dy^i dy^j\,, \tag{5.2.5}$$

while U(1) gauge field is being decomposed as

$$A(\rho) = a d\rho + A_i dx^i\,. \tag{5.2.6}$$

Clearly we can rewrite the above metric (5.2.3) in the above form (5.2.5) with the choice of $N^i = 0$. We consider this as our gauge choice. The inverse metric of (5.2.5) is given by the following expressions

$$g^{\rho\rho} = \frac{1}{N^2}\ ; \quad g^{\rho i} = -\frac{N^i}{N^2}\ ; \quad g^{ij} = \gamma^{ij} + \frac{N^i N^j}{N^2}\,. \tag{5.2.7}$$

The Ricci scalar, on decomposition (5.2.5), is given as follows

$$R(g) = R(\gamma) + K^2 - K_{ij}K^{ij} + \nabla_\mu(-2Kn^\mu + 2n^\nu \nabla_\nu n^\mu)\,, \tag{5.2.8}$$

where K_{ij} is extrinsic curvature. It is given by the following expression

$$\begin{aligned} K_{ij} &= \frac{1}{2N}(\dot{\gamma}_{ij} - \Gamma_{\text{lapse}-\text{shift}}) \\ \Gamma_{\text{lapse}-\text{shift}} &= D_i N_j - D_j N_i\,, \end{aligned} \tag{5.2.9}$$

and the unit normal vector to the constant ρ hypersurface is given by $n^\mu = \left(\frac{1}{N}, -\frac{N^i}{N}\right)$. Also here the covariant derivative with respect to the induced metric γ_{ij} given by D_i. The action from (5.2.1), reduced in radial direction and normal direction, is given by

$$\begin{aligned} S &= \int d^4y N\sqrt{\gamma}[R(\gamma) + K^2 - K^{ij}K_{ij} - \frac{1}{2N^2}(D_\rho\phi - N^i D_i\phi)^\dagger(D_\rho\phi - N^j D_j\phi) \\ &- \frac{m^2}{2}\phi^\dagger\phi - \gamma^{ij}(D_i\phi)^\dagger D_j\phi \\ &- \frac{1}{2}\frac{\gamma^{ij}}{N^2}\left(F_{\rho i} - N_k F_i^k\right)\left(F_{\rho j} - N_m F_j^m\right) \\ &- \frac{1}{2}\frac{\gamma^{ij}}{N^2}\left(W_{ri} - N_k W_i^k\right)\left(W_{rj} - N_m W_j^m\right) \\ &- \frac{1}{4}F_{ij}F^{ij} - \frac{1}{4}W_{ij}W^{ij}] \\ &- \kappa\epsilon^{rijkm}\left(\partial_\rho A_i - \partial_i A_\rho - N_k F_i^k\right)W_{jk}B_m + \int_{\partial M}\sqrt{-\gamma}d^4y 2K\,. \end{aligned} \tag{5.2.10}$$

Note that canonical momenta conjugate to the fields N, N^i , and a vanish since the corresponding radial derivatives do not appear in the above Lagrangian (5.2.10). We can evaluate the Legendre transform of the Lagrangian to obtain the radial Hamiltonian, given by

$$H = \int d^d y \left(\pi_{ij}\dot{\gamma}^{ij} + \pi_\phi^{a\dagger}\pi_\phi^a + \pi_B^a\dot{B}^a + \pi^i\dot{A}_i\right) - L\,. \tag{5.2.11}$$

Here by radial Hamiltonian we mean that canonically conjugate momenta to the fields are taken as derivative of Lagrangian (5.2.10) w.r.t radial derivative of the fields, as we did in (2.6.6). One can show that

$$H = \int d^d y \left(N\mathcal{H} + \mathcal{N}^i \mathcal{H}^i + a\mathcal{F} \right) , \tag{5.2.12}$$

where

$$\begin{aligned}
\mathcal{H} &= -\frac{1}{\sqrt{-\gamma}} \left(\gamma_{ik}\gamma_{jl} - \frac{1}{d-1}\gamma_{ij}\gamma_{kl} \right) \pi^{ij}\pi^{kl} - 2\pi_{\phi}^{\dagger a}\pi_{\phi}^{a} - \frac{1}{2}\gamma_{ij}\pi_B^i\pi_B^j \\
&- \frac{1}{2}\gamma_{ij} \left(\pi_A^i + \frac{\kappa}{2}\epsilon^{\rho inkm} W_{nk} B_m \right) \left(\pi_A^j + \frac{\kappa}{2}\epsilon^{\rho in'k'm'} W_{n'k'} B_{m'} \right) \\
&+ \sqrt{-\gamma}\{ \gamma^{ij} (D_i\phi^a)^\dagger (D_j\phi^a) + R + 12 \\
&+ \frac{1}{4}F_{ij}F^{ij} + \frac{1}{4}W_{ij}^a W^{a\,ij} + \frac{m^2}{2}\phi^{a\dagger}\phi^a \} ,
\end{aligned} \tag{5.2.13}$$

$$\begin{aligned}
\mathcal{H}^i &= D_j\pi^{ij} + (\pi_\phi^\dagger D^i\phi^a + \pi_\phi^a D^i\phi^\dagger) + F^{ij}(\pi_A^s + \frac{\kappa}{2}\epsilon^{\rho snkm} W_{nk} B_m)\gamma_{sj} \\
&+ \frac{1}{2} W^{ij,a}\pi_{B,m}^a \gamma^{jm} ,
\end{aligned} \tag{5.2.14}$$

where π^{ij} correspond to canonically conjugate momentum of the metric γ^{ij}, π_B and π_A correspond to canonical conjugate momentum of SU(2) gauge field B and U(1) gauge field A and π_ϕ is the conjugate momentum to ϕ. Also

$$\mathcal{F} = D_i \pi^i . \tag{5.2.15}$$

Finally the Hamilton's equation (from (5.2.12)) of motion imply

$$\mathcal{H} = \mathcal{H}^i = \mathcal{F} = 0 . \tag{5.2.16}$$

Following [51], it can be shown that. the momentum constraint $\mathcal{H}^i = 0$ and Gauss's law constraint $\mathcal{F} = 0$ represent transverse diffeomorphism invariance and gauge invariance of the on shell action. We will focus on Hamiltonian constraint, $\mathcal{H} = 0$, which implies invariance of on shell action under radial diffeomorphism. Note $\mathcal{H}$ can be written as

$$\mathcal{H} = \{S, S\} - \mathcal{L}_d\,, \tag{5.2.17}$$

where $\mathcal{L}_d$ is the d dimensional part of the Lagrangian (d corresponds to boundary dimension) and the bracket expressed as:

$$\begin{aligned} \{S, S\} &= -\frac{1}{\sqrt{-\gamma}}[\left(2\gamma_{ik}\gamma_{jl} - \frac{1}{d-1}\gamma_{ij}\gamma_{kl}\right)\pi^{ij}\pi^{kl} - 2\pi^a_{\phi^\dagger}\pi^a_\phi - \frac{1}{2}\gamma_{ij}\pi^i_B\pi^j_B \\ &- \frac{1}{2}\gamma^{ij}\left(\pi^A_i + \frac{\kappa}{2}\epsilon^{\rho inkm}W_{nk}B_m\right)\left(\pi^A_j + \epsilon^{\rho jn'k'm'}W_{n'k'}B_{m'}\right)]\,. \end{aligned} \tag{5.2.18}$$

On substituting $\mathcal{H} = 0$ we obtain the flow equation

$$\{S, S\} = \mathcal{L}_d\,. \tag{5.2.19}$$

We mentioned in section 2.6 in equation (2.6.17) (as proposed by Verlinde [54]), the on shell action can be written as a local and non local part,

$$S = S_{loc} + \Gamma\,, \tag{5.2.20}$$

where Γ is the quantum effective action. As prescribed before, the on shell action can have derivative decomposition as follows

$$\begin{aligned} S_{loc} &= S^{(0)}_{loc} + S^{(2)}_{loc} + \ldots\ldots\ldots\ldots \\ S^{(0)}_{loc} &= \int \sqrt{-\gamma}W\left(\phi^\dagger(\rho)\phi(\rho)\right) \\ S^{(2)}_{loc} &= \int \sqrt{-\gamma}\left\{\Phi\left(\phi^\dagger(\rho)\phi(\rho)\right)R_d + M\left(\phi^\dagger(\rho)\phi(\rho)\right)\left((D_i\phi(\rho))^\dagger(D_i\phi(\rho))\right)\right\} \\ &, \end{aligned} \tag{5.2.21}$$

where in the above expressions, by the superscripts we mean the number of derivatives/weights and R_d is d dimensional Ricci scalar. We carry out most general expansion of S_{loc} which was considered in ([51], [54]). For d dimensional Lagrangian we write

$$\begin{aligned}\mathcal{L}_d^{(0)} &= \left\{d(d-1) - \frac{m^2}{2}\left(\phi^\dagger(\rho)\phi(\rho)\right)\right\} \\ \mathcal{L}_d^{(2)} &= \left\{R_d + \gamma^{ij}\left((D_i\phi(\rho))^\dagger(D_j\phi(\rho))\right)\right\} \\ \mathcal{L}_d^{(4)} &= -\frac{1}{4}F_{ij}F^{ij} - \frac{1}{4}W_{ij}W^{ij}\,, \end{aligned} \tag{5.2.22}$$

where the expression $D_i\phi(\rho)$ is d dimensional covariant derivative of ϕ. This is according to our ansatz(5.2.3,5.2.4), given by

$$(D_i\phi)^a(D_i\phi)^a = \left(w(\rho)^2 e^{-2x_2(\rho)} - \frac{a^2(\rho)}{g(\rho)}\right)\phi^\dagger(\rho)\phi(\rho)\,, \tag{5.2.23}$$

where $a(\rho)$ is the time component of $U(1)$ gauge field and $w(\rho)$ is the ω_2 component of $SU(2)$ gauge field (5.2.4). Also Γ is the quantum effective action contains higher derivative and nonlocal terms. So the Hamilton -Jacobi equation actually reduces to the following equations (5.2.19,5.2.21,5.2.22)

$$\left\{S_{loc}^{(0)}, S_{loc}^{(0)}\right\} = \mathcal{L}_d^{(0)} \tag{5.2.24}$$

$$2\left\{S_{loc}^{(0)}, S_{loc}^{(2)}\right\} = \mathcal{L}_d^{(2)} \tag{5.2.25}$$

$$\left\{S_{loc}^{(2)}, S_{loc}^{(2)}\right\} + 2\left\{S_{loc}^{(4)}, \Gamma\right\} = \mathcal{L}_d^{(4)}\,. \tag{5.2.26}$$

We express the radial derivative of the fields with Hamiltonian equation of motion

$$\frac{d\phi}{dr} = \frac{\partial H}{\partial \pi_\phi} = \frac{\partial H}{\partial(\frac{\partial S}{\partial \phi^\dagger})}$$

where by S we mean the on Shell action. So inserting the expansions(5.2.21) and using the fact that at on shell, the momenta corresponding to a field can be expressed as the derivative of the on shell action w.r.t the field (2.6.12), we can write ([52],

[51], [53])

$$
\begin{aligned}
\frac{d\phi}{d\rho} &= \frac{\partial W}{\partial \phi} \\
\frac{dw(\rho)}{d\rho} &= \frac{\partial (M(\phi^\dagger\phi)(D_i\phi)^\dagger(D^i\phi)}{\partial w(\rho)} = M(\phi^\dagger\phi)\, Exp(-2x_2(\rho))\, w(\rho)\, \phi^\dagger\phi \\
\frac{da(\rho)}{d\rho} &= \frac{\partial (M(\phi^\dagger\phi)(D_i\phi)^\dagger(D^i\phi)}{\partial a(\rho)} = -M(\phi^\dagger\phi)\, \frac{1}{g(\rho)}\, a(\rho)\, \overline{\phi}\phi\,, \qquad (5.2.27)
\end{aligned}
$$

and finally we write for the gravity part following [53]

$$
\begin{aligned}
\frac{\partial g_{\mu\nu}}{\partial \rho} g^{\mu\nu} &= (-2\pi_{\mu\nu} + \frac{2}{d-1}\pi^\lambda_\lambda g_{\mu\nu})g^{\mu\nu} \\
&= \frac{2}{d-1}\pi^\lambda_\lambda \\
&= g^{\mu\nu}\frac{2}{d-1}\frac{\partial}{\partial g_{\mu\nu}}\{\int \sqrt{g}W(\phi)\} \\
\Rightarrow \frac{dg_{\mu\nu}}{d\rho} &= -\frac{W(\overline{\phi}\phi)}{d-1} g_{\mu\nu} + C(\rho)\,, \qquad (5.2.28)
\end{aligned}
$$

where we introduce the constant $C(\rho)$. Note that $C(\rho)$ is independent of $g_{\mu\nu}$ but it can differ for different component of metric. It is better to say that it actually gives anisotropy of the metric around the fixed point. Since we want to consider the simplest solution we will set $C = 0$ for all metric.

Recall we obtained the field derivatives from the expressions $W(\phi^\dagger\phi), M(\phi^\dagger\phi)$ in (5.2.27,5.2.28). Hence we will effectively treat the said expressions as potential and call them by potential in the rest of the chapter. It implies, from (5.2.28)

$$
\frac{U'}{U} = \frac{W(\phi^\dagger\phi)}{d-1} \quad ; \quad x_i' = -\frac{W(\phi^\dagger\phi)}{2(d-1)} \quad ; \quad \frac{g'}{g} = -\frac{W(\phi^\dagger\phi)}{d-1}. \qquad (5.2.29)
$$

As expressed in (5.2.26), to write generalized expression for these brackets, following [51], [52] we will assign weight to all fields, as we did in chapter 2.6. This is as following:

$$
\begin{aligned}
\gamma_{\mu\nu}, \phi, \Gamma \quad &: 0 \\
\partial_\mu, B_\mu, A_\mu \quad &: 1 \\
R, R_{\mu\nu}, R_{\mu\nu\rho\sigma}, \partial^2 \quad &: 2 \\
\frac{\partial}{\partial a(\rho)}, \frac{\partial}{\partial w(\rho)} \quad &: d-1 \\
\frac{\partial}{\partial h_{\mu\nu}(\rho)}, \frac{\partial}{\partial \phi(\rho)} \quad &: d
\end{aligned} \tag{5.2.30}
$$

$$
S_{loc;w-d} \;=\; \int d^d y \sqrt{-\gamma} L^{(w)} \,. \tag{5.2.31}
$$

Recall in (5.2.26) we wrote the equations upto weight 4. Similarly one can construct weight d equation, where d is the boundary dimension, as

$$
2\left\{S_{loc}, \Gamma\right\}_d = -\left\{S_{loc}, S_{loc}\right\}_d + \mathcal{L}_d^{(d)} \,, \tag{5.2.32}
$$

where $\{S_{loc}, \Gamma\}_d$ and $\{S_{loc}, S_{loc}\}_d$ represents weight d term from the bracket. Also by the term $\mathcal{L}_d^{(d)}$ we mean weight d term of $\mathcal{L}_d$. Finally the expression (5.2.32) for generalized bracket is

$$
\begin{aligned}
& \frac{W\left(\phi^\dagger(\rho)\phi(\rho)\right)}{d-1}\gamma^{kl}\frac{\partial\Gamma}{\partial\gamma^{kl}} - \frac{\partial W\left(\phi^\dagger(\rho)\phi(\rho)\right)}{\partial\phi^\dagger}\frac{\partial\Gamma}{\partial\phi} - \frac{\partial W\left(\phi^\dagger(\rho)\phi(\rho)\right)}{\partial\phi}\frac{\partial\Gamma}{\partial\phi^\dagger} \\
- \;& \gamma^{ij}\frac{\partial S_{loc;2-d}}{\partial B^i}\frac{\partial\Gamma}{\partial B^j} - \gamma^{ij}\frac{\partial S_{loc;2-d}}{\partial A^i}\frac{\partial\Gamma}{\partial A^j} \\
= \;& \frac{1}{2}(\mathcal{L}_d^{(d)}) - \frac{1}{2}\left\{S, S\right\}_d + \frac{1}{2}\kappa\epsilon^{\rho ijkm}W_{jk}B_m\frac{\partial S_{loc}}{\partial A^i} \,.
\end{aligned} \tag{5.2.33}
$$

After some rearrangement of the above equation, we write Callan-Symmanzik equation as

$$
\begin{aligned}
& \gamma^{kl}\frac{\partial\Gamma}{\partial\gamma^{kl}} - \beta_\phi\frac{\partial\Gamma}{\partial\phi} - \beta_{\phi^\dagger}\frac{\partial\Gamma}{\partial\phi^\dagger} - \beta_B\frac{\partial\Gamma}{\partial B^j} - \beta_{a(\rho)}\frac{\partial\Gamma}{\partial A^j} \\
= \;& \frac{1}{2}\left\{\frac{W\left(\phi^\dagger(\rho)\phi(\rho)\right)}{d-1}\right\}^{-1}\left[(\mathcal{L}_d^{(d)}) - \left\{S, S\right\}_d + \kappa\epsilon^{\rho ijkm}W_{jk}B_m\frac{\partial S_{loc}}{\partial A^i}\right] \,,
\end{aligned} \tag{5.2.34}
$$

with

$$
\begin{aligned}
\beta_\phi &= \left\{\frac{W\left(\phi^\dagger(\rho)\phi(\rho)\right)}{d-1}\right\}^{-1}\frac{\partial W\left(\phi^\dagger(\rho)\phi(\rho)\right)}{\partial\phi^\dagger}\\
\beta_{\phi^\dagger} &= \left\{\frac{W\left(\phi^\dagger(\rho)\phi(\rho)\right)}{d-1}\right\}^{-1}\frac{\partial W\left(\phi^\dagger(\rho)\phi(\rho)\right)}{\partial\phi}\\
\beta_{a(r)} &= \left\{\frac{W\left(\phi^\dagger(\rho)\phi(\rho)\right)}{d-1}\right\}^{-1}\gamma^{ij}\frac{\partial S_{loc;2-d}}{\partial A^i}\\
\beta_{w(r)} &= \left\{\frac{W\left(\phi^\dagger(\rho)\phi(\rho)\right)}{d-1}\right\}^{-1}\gamma^{ij}\frac{\partial S_{loc;2-d}}{\partial B^i}\,.
\end{aligned}
\tag{5.2.35}
$$

We define the β functions as above. The fixed point is given by simultaneous zero of the β functions. Clearly when we have $\beta_\phi = \beta_{\phi^\dagger} = 0$ this implies, the potential term $W\left(\phi^\dagger(r)\phi(r)\right)$ is at its extrema (5.2.35). In order to understand it for SU(2) gauge field and vector field, let us recall (5.2.31,5.2.21); the vanishing of $\beta_{a(r)}$ and $\beta_{w(r)}$ needs, for nonzero ϕ,

$$
M\left(\phi^\dagger(\rho)\phi(\rho)\right) = 0\,, \tag{5.2.36}
$$

at fixed point. We will discuss these points further in next sections. So far we have considered Callan Symanzik equation which is weight d part of $\mathcal{H} = 0$ constraint. Next we consider weight 0 and weight 2 part of the same constraint. These latter equations, which are equations for $W(\phi^\dagger\phi), M(\phi^\dagger\phi)$ as introduced in (5.2.27,5.2.28), are direct consequence of the constraint $\mathcal{H} = 0$. These equations can be obtained from (5.2.19) where the canonically conjugate momenta is expressed as the derivative of on shell action as given in (2.6.12) and we have the on shell action, which is expanded as (5.2.21). We will call them potential equations which are as follows. The potential equation at zeroeth derivative order is given by

$$
\begin{aligned}
\frac{d}{4(d-1)}[W(\phi^\dagger\phi)]^2 &- \frac{\partial W(\phi^\dagger\phi)}{\partial\phi^\dagger}\frac{\partial W(\phi^\dagger\phi)}{\partial\phi} - \frac{1}{2}\phi^\dagger\phi\, M(\phi^\dagger\phi)W(\phi^\dagger\phi)\left(e^{-2x_2} - \frac{1}{g(\rho)}\right)\\
&= V(\phi^\dagger\phi)\,.
\end{aligned}
\tag{5.2.37}
$$

At two derivative level we have the expression as

$$
\begin{aligned}
-\frac{1}{2}(D_i\phi)^\dagger(D^i\phi)+R_d &= \frac{d-2}{d-1}\left[\Phi(\phi)\,R_d+M(\phi^\dagger\phi)(D_i\phi)^\dagger(D^i\phi)\right]W(\phi^\dagger\phi)\\
&- \left\{\frac{\partial W(\phi^\dagger\phi)}{\partial\phi^\dagger}\frac{\partial\Phi(\phi^\dagger\phi)}{\partial\phi}\right\}R_d\\
&- \left\{\frac{\partial W(\phi^\dagger\phi)}{\partial\phi^\dagger}\frac{\partial M(\phi^\dagger\phi)}{\partial\phi}\right\}(D_i\phi^\dagger)(D^i\phi)\\
&+ M(\phi^\dagger\phi)\frac{\partial W(\phi^\dagger\phi)}{\partial\phi^{a\,\dagger}}\left\{-\frac{a(\rho)}{g(\rho)}(D_i\phi)^a+f^{bca}(D_i\phi)^bB^c(r)\right\}\\
&- W(\phi^\dagger\phi)M(\phi^\dagger\phi)\left\{-\frac{a(\rho)^2}{g(\rho)}+e^{-2x_2(\rho)}w(\rho)^2\right\}\\
&- \frac{1}{2}M(\phi^\dagger\phi)e^{-4x_2(\rho)}[\phi^\dagger\phi]^4]\\
&- \frac{1}{2}\{\left(-\frac{1}{g(\rho)}(\phi^\dagger\phi)^2M(\phi^\dagger\phi)+\frac{\kappa}{2}\epsilon^{\rho inkm}(W_{nk}b_m)\right)\times\\
& (-\frac{1}{g(\rho)}(\phi^\dagger\phi)^2M(\phi^\dagger\phi))\\
&+ \frac{\kappa}{2}\epsilon^{\rho in'k'm'}(W_{n'k'}b_{m'}))\}\,. \qquad (5.2.38)
\end{aligned}
$$

5.3 Critical Points : AdS fixed point

In previous section we have described the critical points which are given by simultaneous zero of β functions. From (5.2.35), we note that the first condition for zero of the β function is, the extrema of $W(\phi^\dagger\phi)$ has to vanish. To find the critical points, we have to solve the potential equation (5.2.37) order by order. We expand $W(\phi^\dagger\phi)$ perturbatively as

$$W(\phi^\dagger\phi)=W_o+W_1(\phi^\dagger\phi)+W_2(\phi^\dagger\phi)^2+.....\,, \qquad (5.3.1)$$

where W_o, W_1, W_2 are constant coefficient terms. Clearly when we extremize the above (5.3.1), w.r.t. ϕ, one trivial fixed point we find which is at $\phi=0$. Combining,(5.2.21), (5.2.23), (5.2.31), (5.2.35), (5.3.1), one can found that $\phi=0$ is also the zero of the β function which is associated with SU(2) gauge field and U(1) gauge field. Moreover one can also choose, at this fixed point SU(2) gauge field $w(\rho)=0$. Then from the action (5.2.1) it is clear that remaining action describes gravity coupled to U(1) gauge field. In [74] it was shown that such action has RN AdS black hole solution. According to [74], AdS RN black hole has a near horizon geometry given by $AdS_2 \times R^3$ which along with some deformation, is dual to the metallic phase. So it is suggestive that our first fixed point does have correspondence to metallic phase. Substituting (5.3.1), we solve the potential equation (5.2.37) order by order. It appears that first order term is

$$W_o = 2(d-1)\,. \tag{5.3.2}$$

Before considering the next order term, we consider the term $\frac{1}{2}\phi^\dagger\phi\, M(\phi^\dagger\phi)W(\phi^\dagger\phi)\left(e^{-2x_2(\rho)} - \frac{1}{g(\rho)}\right)$ from (5.2.37). Near AdS background $e^{-2x_2(\rho)} = \frac{1}{g(\rho)} = \frac{1}{\rho^2}$ (We consider AdS radius $L = 1$). So exactly near AdS fixed point the contribution of this term term will be zero. The coefficient equation in first order of $\phi^\dagger\phi$ is given by

$$\frac{d}{4(d-1)}(2W_oW_1) - 2W_1^2 = -\frac{1}{2}m^2\,. \tag{5.3.3}$$

So we write

$$W_1 = \frac{1}{2}\triangle_\pm\,. \tag{5.3.4}$$

Substituting (5.3.4) in (5.3.3) gives

$$\frac{d}{2}\triangle_\pm - \frac{1}{2}\triangle_\pm^2 = -\frac{1}{2}m^2\,. \tag{5.3.5}$$

Near AdS fixed point, since $\phi = 0$, the quadratic and higher order terms in ϕ can be ignored. Therefore upto quadratic expression the potential $W(\phi^\dagger\phi)$ can be written as

$$W(\phi^\dagger\phi) = 2(d-1) + \frac{1}{2}\triangle_\pm\phi^\dagger\phi\,, \tag{5.3.6}$$

where $\triangle_\pm$ is the dimension of dual operator. Near AdS fixed point the expression of the scalar field can be obtained by integrating ϕ derivative part of (5.2.27)

$$\phi(\rho) = A_1\rho^{\triangle_+} + A_2\rho^{\triangle_-}\ ;\ \triangle_\pm = \frac{d \pm \sqrt{d^2+4m^2}}{2}\,. \tag{5.3.7}$$

In order to find an approximate expression of $w(\rho), a(\rho)$ near fixed point, we recall the equation (5.2.27),

$$\begin{aligned} \frac{dw(\rho)}{d\rho} &= M(\phi^\dagger\phi)\, Exp(-2x_2(\rho))\, w(\rho)\, \phi^\dagger\phi = \frac{c_1}{\rho^2}(\phi^\dagger\phi-\alpha)(\phi^\dagger\phi)w(\rho) \\ \frac{da(\rho)}{d\rho} &= -M(\phi^\dagger\phi)\, \frac{1}{f(\rho)}\, a(\rho)\, \phi^\dagger\phi = \frac{c_1}{\rho^2}(\phi^\dagger\phi-\alpha)(\phi^\dagger\phi)a(\rho)\,. \end{aligned} \tag{5.3.8}$$

In the above we have substituted the expression of Ads background metric

$$Exp(-2x_2(\rho)) = \frac{1}{g(\rho)} = \frac{1}{\rho^2}$$

Also we use the expression $M(\phi^\dagger\phi) = \phi^\dagger\phi - \alpha$, which actually we have established in the next section (5.4.3). Integrating (5.3.8) one can obtain the series expression

$$\begin{aligned} ln[w(\rho)] &= -ln[a(\rho)] = \frac{c_1}{\rho}\{\alpha(A_1+A_2)^2 - (A_1+A_2)^4\} \\ &+ c_1 ln\rho\{-2\alpha(A_1+A_2)(A_1\frac{\triangle_+}{2} \\ &+ A_2\frac{\triangle_-}{2}) + 3(A_1+A_2)^3(A_1\frac{\triangle_+}{2} + A_2\frac{\triangle_-}{2})\} \\ &+ \,.......\,, \end{aligned} \tag{5.3.9}$$

where A_1, A_2 are integration constant. One can derive similar expression for $a(\rho)$.

5.4 Nontrivial fixed point

We have mentioned in the earlier section that nontrivial fixed point appears with two conditions; extrema of $W(\phi^\dagger\phi)$ is zero and $M(\phi^\dagger\phi)$ is zero. In order to study the theory explicitly, we recall the perturbative expansion of $W(\phi^\dagger\phi)$ which we keep upto quadratic term in $\phi^\dagger\phi$

$$W(\phi^\dagger\phi) = W_o + \frac{1}{2}\triangle_\pm\phi^\dagger\phi + c_1(\phi^\dagger\phi)^2 + ...$$

Extremizing $W(\phi^\dagger\phi)$, we find, one extrema lies at $\phi^\dagger = 0$ (which corresponds to

AdS fixed point) and the other one corresponds to

$$\frac{1}{2}\triangle_{\pm} + 2c_1\phi^{\dagger}\phi = 0\,. \tag{5.4.1}$$

From the above we find that nontrivial fixed point lies on a circle $\phi^{\dagger}\phi = \alpha$ with

$$\alpha = -\frac{\triangle_{\pm}}{4c_1}\,. \tag{5.4.2}$$

First we note that at nontrivial fixed point, the vacuum lies on a circle $\phi^{\dagger}\phi = \alpha$, which implies $\phi = e^{i\chi(x)}, \phi^{\dagger} = e^{-i\chi(x)}$, where χ is any function of space-time variable x and α is given by (5.4.2). Clearly, since the solution has U(1) symmetry, this spontaneously breaks $SU(2) \times U(1)$ symmetry of the theory to $U(1) \times U(1)$. Following [36] this breakdown of symmetry $SU(2) \to U(1)$ with nontrivial v.e.v of the scalar, in a model of gravity coupled to SU(2) and U(1) gauge field, gives rise to antiferromagnetism. Since, we have shown that our model is showing a similar feature so it is suggestive that this nontrivial fixed point corresponds to antiferromagnetic phase in the dual theory.

We mentioned earlier (5.2.36), that when the scalar field ϕ takes a nontrivial v.e.v, the zero of the β function implies the condition, potential at two derivative order $M(\phi^{\dagger}\phi) = 0$. This requires $M(\phi^{\dagger}\phi)$, to have a series form around the nontrivial fixed point

$$M(\phi^{\dagger}\phi) = (\phi^{\dagger}\phi - \alpha) + O((\phi^{\dagger}\phi - \alpha)^2) + \ldots\,, \tag{5.4.3}$$

where α is the value of v.e.v of $\langle\phi^{\dagger}\phi\rangle$ at the nontrivial fixed point (5.4.2). Next we consider the potential equation at zeroeth order (5.2.37), In that equation, we consider the fact $e^{-2x_2(\rho)} \neq \frac{1}{g(\rho)} \neq \frac{1}{\rho^2}$. We denote at the nontrivial fixed point

$$e^{-2x_2(\rho)} - \frac{1}{g(\rho)} = c + O(\phi^{\dagger}\phi) + \ldots.\,, \tag{5.4.4}$$

where c is the zeroeth order term of the above expansion. Substituting this (5.4.4) in (5.2.37) and also using (5.4.3) we obtain the zeroeth term of $W(\phi^{\dagger}\phi)$ which is

$W_o = 2(d-1)$ and the term of the order $\phi^\dagger\phi$ is given by

$$d\triangle_\pm - \triangle_\pm^2 = -m^2 - 2c\alpha(d-1)\,. \tag{5.4.5}$$

We see in the above expression that once we set $c = 0$, we get back the AdS RN expression (5.3.5) which also gives AdS RN geometry near critical point. So we have the equation for $\triangle_\pm$ is

$$\triangle_\pm^2 + \left[\frac{c}{2c_1}(d-1) - d\right]\triangle_\pm - m^2 = 0\,. \tag{5.4.6}$$

$$\begin{aligned} \triangle_\pm &= -\frac{b}{2} \pm \frac{1}{2}\sqrt{b^2 + 4m^2} \\ b &= \left[\frac{c}{2c_1}(d-1) - d\right]\,. \end{aligned} \tag{5.4.7}$$

Finally we want to comment that since the nontrivial fixed point breaks $SU(2) \times U(1) \to U(1) \times U(1)$, from [36], it is suggestive that the respective fixed point describes an antiferromagnetic phase in the dual theory.

5.5 Instability of metallic phase

In holographic duality, the IR limit of a theory is described by the near horizon limit. In this section we consider a solution with $\text{AdS}_2 \times R^3$ as the near horizon limit. In addition, we have added deformations following [74] to it given by:

$$\begin{aligned} U &= 12\rho^2(1 + u_0\rho^\delta)\ ;\ x_i = v_o(1 + x_{i1}\rho^\delta)\ ;\ a = 2\sqrt{6}\rho(1 + a_0\rho^\delta) \\ w &= w_o\rho^\delta\ ;\ \phi = \phi_o\rho^\delta\ ;\ g = 12\rho^2(1 + g_0\rho^\delta)\,. \end{aligned} \tag{5.5.1}$$

In [74] a similar configuration along with deformation is considered, which represents metallic state at zero temperature [116].

We determine the exponents in (5.5.1) using equations of motion. They come in pairs corresponding to the power of two roots of the equations. We find two marginal operator with $\delta_\pm = 0$ which corresponds to rescaling of x_1, x_2, x_3. The four remaining operator are given by positive weight

$$\begin{aligned} \delta_+ &= 1 \ ; \ -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{12}p^2 e^{-2x_0} - \frac{\kappa}{\sqrt{6}} p e^{-2x_0}} \ ; \\ & -\frac{1}{2} + \sqrt{\frac{1}{12}(1+m^2)} \ ; \ \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{3}p^2 e^{-2x_0}} \,, \end{aligned} \tag{5.5.2}$$

if

$$2\sqrt{6}\kappa < p e^{-x_0} \ ; \ m^2 > 2. \tag{5.5.3}$$

One may observe that all four modes are irrelevant with $\delta_+ > 0$. We see that when both the inequalities in (5.5.3) are satisfied, the solution is stable and following [74] we identify this with metallic state. When the second inequality is violated the respective deformation mode becomes relevant. However in this case the translation symmetry remain conserved since we have $w \to 0$ as $r \to 0$ and the respective deformation which causes the helical structure to arise, is being turned off. When the first inequality becomes violated in (5.5.3), it will cause $\delta_w < 0$ about IR fixed point, make the dual operator relevant. When the deformation associated with $w(r)$ turns out to be relevant, w will take nonzero value at horizon. Consequently the translational symmmetry will break and the respective instability mode will turn on the helical geumetry. In addition, it is also expected that ϕ will also be turned on. We expect, then such relevant modes will drive the IR geometry from $AdS_2 \times R^3$ along with some deformation, to the one with helical symmetry and non-zero ϕ. Hence once this condition is violated, following [74], we understand that system will flow away from metallic phase to another fixed point phase breaking $SU(2) \to U(1)$ which we have seen, suggestive to be antiferromagnetic phase.

5.6 Conclusion

Here we have developed RG formalism for our proposed model, which is SU(2)$\times$ U(1) gauge theory coupled to scalar in adjoint representation in the presence of Chern Simons and Gibbons Hawking term. We derived the Callan Symanzik equation, β functions in this model. We found that the critical points, which is obtained by setting all the β functions simultaneously to zero, corresponds to the extrema of the potential $W(\phi^\dagger\phi)$. We observed the trivial fixed point corresponds to an action which has AdS RN black hole solution, Here the near horizon geometry along with some deformation corresponds to metallic phase. We also noted the fact that our model has a nontrivial fixed point where SU(2)$\times$ U(1) gauge symmetry breaks down to U(1)$\times$ U(1) symmetry and resembles antiferromagnetism. We have also checked the fact that the metallic phase develops instability for certain parametric condition, flows towards another phase which is suggestive to be antiferromagnetic.

Chapter 6

Hyperscaling violating geometry and thermoelectric properties

6.1 Introduction

So far we have restricted our study of different phases of holographic superconductors. In this chapter we would like to deviate a little bit and like to study transport properties. As we mentioned earlier, there are materials discovered, which show anomalous transport behaviours. In particular, there is strange metal phase which shows linear temperature dependence of resistivity. These behaviours are substantially different from the Fermi liquid and understanding such deviations is the one of the motivations for studying this. Since it has been suggested in [100] that hyperscaling violating geometries may be the appropriate set up to look for such phenomena, we consider a different gravity model in this chapter.

We consider hyperscaling violating Lifshitz geometries for this purpose. Such geometries are characterised by two parameters v and ξ, corresponding to Lifshitz scaling and the hyperscaling violation respectively. As mentioned earlier, transport properties for such theories have been discussed in [86–90]. To obtain such geometry one may consider four dimensional Einstein-Maxwell-Axion-Dilaton theory with two $U(1)$ gauge fields. One gauge field is required to introduce Lifshitz like behaviour, while the other plays the role of electromagnetic field. In the present chapter we will be considering transport properties that ensued from this solution and look for scaling behaviour of various transport coefficients.

Our approach will be considering linearised fluctuations around black hole solution. Solving equations of the fluctuations we obtain the transport coefficints. As we explained in chapter 2 this appraoch is amenable for incorporation of different boundary conditions as well as identification of the boundary observables. It has been discussed for electrically charged black hole [88] with hyperscaling violating Lifshitz geometry. In our work, we introduce a magnetic field and obtain a dyonic black hole solution. With this dyonic black hole as the background we study the linear fluctuations. Considering expansion of these fluctuations in orders of frequency, we obtain the solutions for lower orders. The magnetic field enable us to study the magnetic proerties as well, such as Hall angles. We will obtain and discuss the result for thermoelectric coefficients for Dirichlet boundary condition but we can incorporate other boundary conditions as well, as mentioned above in a straightforward manner. The case of electrical black hole may be obtained at the limit of vanishing magnetic field. The expressions are quite invovled and we have taken limits of parameters to discuss scaling properties of the coefficients.

This chapter is structured as follows. In the next section we introduce the asymptotically Lifshitz hyperscaling violating solution. In the next section we introduce the fluctuations in metric and gauge fields, consider their linearised equations of motion and obtain solution in low frequency limit. In section 6.4 we compute the thermoelectric coefficients and discuss their temperature dependence. We conclude in section 6.5. Some of the materials related to the necessary canonical transformation of the fields has been discussed in the appendix B.

6.2 Hyperscaling violating Lifshitz Black Hole

In the present section we derive asymptotically Lifshitz hyperscaling violating solution of the equations of motion from our considered model, in the presence of background magnetic field. We will use this solution as the background. The electrically charged solution and the dyonically charged solution have been described in [86, 88] and [87, 89] respectively, as classical solutions of an Einstein-Maxwell-dilaton-axion system. As mentioned above, it requires 2 gauge fields to obtain the geometry. Here we have coupled them through a symmetric invertible matrix Σ_{IJ}, $I, J = 1, 2$ with positive eigenvalues, where Σ depends on the dilaton field ϕ. In order to violate the momentum conservation, which is required for direct conductivity we include two axion fields, χ^a, where $a = 1, 2$. We include a dilaton dependent prefactor $Z(\phi)$ for the Axion term in the action (6.2.1).

We consider the four dimensional action

$$S = S_{\text{grav}} + S_{\text{scalar}} + S_{\text{axion}} + S_{\text{U(1)}} + S_{\text{Gibbons-Hawkings}} \tag{6.2.1}$$

where

$$\begin{aligned} S_{\text{grav}} &= \int d^4x\sqrt{-g}R\,, \\ S_{\text{scalar}} &= -\int d^4x\sqrt{-g}[\alpha(\partial\phi)^2 + V(\phi)]\,, \\ S_{\text{axion}} &= -\int d^4x\sqrt{-g}[Z(\phi)(\partial\chi^a)^2]\,, \\ S_{\text{U(1)}} &= -\int d^4x\sqrt{-g}[\Sigma_{IJ}F^I_{\mu\nu}F^{J\mu\nu}]\,, \end{aligned} \tag{6.2.2}$$

and three dimensional boundary action is given by

$$S_{\text{Gibbons-Hawkings}} = \frac{1}{2\kappa^2}\int\limits_{\partial M} d^3x\sqrt{-\gamma}2K\,, \tag{6.2.3}$$

.

where $\kappa^2 = 8\pi G$ and the $S_{\text{Gibbons-Hawkings}}$ is Gibbons-Hawking boundary term. $V(\phi)$ is the potential for dilaton fields. We have derived the equations of motion obtained from (6.2.1) given by

$$R_{\mu\nu} = \alpha\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}V(\phi)g_{\mu\nu} + Z(\phi)\partial_\mu\chi^a\partial_\nu\chi^a + 2\Sigma_{IJ}(\phi)(F^I_{\mu\lambda}F^{J\lambda}_\nu - \frac{1}{4}g_{\mu\nu}F^I_{\rho\sigma}F^{J\rho\sigma}),$$
$$\nabla^\mu(\Sigma_{IJ}(\phi)F^J_{\mu\nu}) = 0,$$
$$\nabla^\mu(Z(\phi)\partial_\mu\chi^a) = 0 \quad \text{and} \quad 2\alpha\Box\phi - V'(\phi) = \Sigma'_{IJ}(\phi)F^I_{\rho\sigma}F^{J\rho\sigma},$$
(6.2.4)

which represnts Einstein, Maxwell, axion and dilaton equations.

We have chosen the following ansätz for the metric, axion and the gauge fields

$$\begin{aligned} ds^2_B &= \gamma_{\mu\nu}dx^\mu dx^\nu = dr^2 + e^{2H}(-g(r)dt^2 + dx^2 + dy^2)\,, \\ \chi^1_B &= kx, \quad \chi^2_B = ky, \quad \phi_B = \phi_B(r), \quad A^I = a^I = a^I_t(r)dt + \frac{B^I}{4}\epsilon_{ab}x^a dx^b \end{aligned}$$
(6.2.5)

to obtain asymptotically Lifshitz hyperscaling violating black hole solution. Here we denote the background metric tensor by γ_{ab} . Note that since we have chosen linear axion in the background, it is breaking the translation invariance. Thus by our choice of background we are incorporating a mechanism to break momentum conservation to obtain DC conductivity. One gauge field gives rise to Lifshitz like behaviour of the metric. The solution has electric and magnetic charge with respect to the second gauge field, though we have kept the constant magnetic field $F^I_{ab} = \frac{1}{2}B^I\epsilon_{ab}$ associated with both the gauge fields.

We substitute the ansatz (6.2.5) in the second equation of (6.2.4). We found that the elctric charges $q_I = -g^{-1/2}e^H\Sigma_{IJ}\partial_r a^J_t$ is constant. On substitution of the ansatz

(6.2.5), the first and the last equation (6.2.4), give rise the following equations:

$$\begin{aligned} \frac{g''}{2g} + 3H'\frac{g'}{2g} - \frac{g'^2}{4g^2} &= k^2 Z(\phi)e^{-2H} + 2e^{-4H}Q, \\ H'' + H'(3H' + \frac{g'}{2g}) + k^2 Z(\phi)e^{-2H} &= \frac{1}{2}V + e^{-4H}Q, \\ (6H'^2 + 4H'\frac{g'}{2g}) &= \alpha(\partial_r\phi)^2 - 2p^2 Z(\phi)e^{-2H} - V - 2e^{-4H}Q, \\ 2\alpha[\partial_r^2\phi + (3H' + \frac{g'}{2g})\partial_r\phi] - V'(\phi) &= 2e^{-4H}Q', \end{aligned} \tag{6.2.6}$$

where $Q = (\Sigma^{IJ}(\phi)q_I q_J + \frac{1}{4}\Sigma_{IJ}(\phi)B^I B^J)$

Provided we have the expression of $Z(\phi)$ and $\Sigma_{IJ}(\phi)$, we can obtain these equations to find out the metric, the dilaton, the Maxwell field and the potential.

Very similarly like the electrically charged black hole, these equations gives an exact dyonic black hole solution [87, 89], which depend on two parameters v and ξ. It is more convenient to choose another radial coordinate z to get appropriate asymptotic behavior and the dyonic solution in terms of v is as follows. The metric is

$$ds^2 = z^{-\xi}[-z^{2v}F(z)dt^2 + \frac{dz^2}{z^2F(z)^2} + z^2(dx^2 + dy^2)], \tag{6.2.7}$$

where we set $e^{2H} = z^{2-\xi}$ in (6.2.5). The blackening factor $F(z)$ is given by

$$\begin{aligned} F(z) &= 1 + \frac{k^2}{(2-\xi)(v-2)z^{2v-\xi}} - \frac{m}{z^{2+v-\xi}} + \frac{8q_2^2}{(2-\xi)(v-\xi)z^{2(v+1-\xi)}} \\ &+ \frac{B^2 z^{2v-6}}{16(4+\xi-3v)(2-v)}. \end{aligned} \tag{6.2.8}$$

This z coordinate is given in terms of r by the relation

$$dr = -sgn(\xi)z^{-\xi/2}\mathcal{F}^{-1/2}(z)\frac{dz}{z}. \tag{6.2.9}$$

Other fields and functions are given following manner: The prefactors, $\Sigma_{IJ}(\phi)$

and $Z(\phi)$ are

$$\Sigma_{11}(\phi) = \frac{1}{4}e^{[(\xi-4)/\mu]\phi}, \quad \Sigma_{22}(\phi) = \frac{1}{4}e^{[(2v-2-\xi)/\mu]\phi}, \quad \Sigma_{12} = 0, \quad Z(\phi) = \frac{1}{2}e^{[\mu/(\xi-2)]\phi}, \tag{6.2.10}$$

where $\alpha = 1/2$ and μ is given by $2\mu^2\alpha = (2-\xi)(2v-2-\xi)$. In terms of new radial coordinate z, the dilaton, the axion and the gauge fields are

$$\begin{aligned} \phi &= \mu \log z, \\ \chi^1 &= kx, \\ \chi^2 &= ky, \\ a_t^1 &= \frac{4sgn(\xi)q_1}{2+v-\xi}(z^{2+v-\xi} - z_h^{2+v-\xi}), \\ a_t^2 &= \frac{4sgn(\xi)q_2}{\xi-v}(z^{\xi-v} - z_h^{\xi-v}), \end{aligned} \tag{6.2.11}$$

where the charge q_1 is given by

$$q_1^2 = (2+v-\xi)(v-1)/8, \tag{6.2.12}$$

and $V(\phi)$

$$V(\phi) = -(2+v-\xi)(1+v-\xi)e^{\xi\phi/\mu} - \frac{2v-2-\xi}{4(v-2)}B^2e^{(\xi+2v-6)(\phi/\mu)}. \tag{6.2.13}$$

represents the potential.

6.3 Fluctuation

Thermoelectric coefficients are expressed in terms of correlation function of operators which are dual to linear fluctuation. Therefore, for computation of the former we need to consider linear fluctuations in the metric and the gauge fields around the background solution. The fluctuation in the metric is given by

$$\gamma_{ij} = \gamma_{Bij} + \sigma_{ij}, \tag{6.3.1}$$

whereas fluctuation in the fields are given by

$$A^I_i = A^I_{Bi} + \alpha^I_i, \quad \phi = \phi_B + \varphi, \quad \chi^a = \chi^a_B + \beta^a, \tag{6.3.2}$$

where i, j takes values on t, x and y. We define $W^j_i = \gamma^{jk}\sigma_{ik}$. To simplify the analysis, we set $W^t_t = W^x_x = W^y_y = W^y_x = 0$ and $\varphi = \alpha^I_t = 0$ consistently. That leaves nonzero fluctuations to be W^a_t, W^t_a, α^I_a and β^a. Note that W^t_a is related to W^a_t and so W^t_a will not be considered separately. Assuming these fields depend only on t and r, the linearised equations for these fluctuations for the background given in the ansatz (6.2.5) are as follows:

$$\begin{aligned}
& [\partial_r^2 + (3\partial_r H - \frac{\partial_r g}{2g})\partial_r - e^{-2H}(2k^2 Z + e^{-2H}\Sigma_{IJ}B^I B^J)]W^a_t \\
= \; & -2e^{-2H}[kZ(\partial_t\beta^a) + 2\Sigma_{IJ}(\partial_r\alpha^I_t)(\partial_r\alpha^J_a) + e^{-2H}\Sigma_{IJ}(\partial_t\alpha^I_b)\epsilon_{ab}B^J], \\
& \partial_r\partial_t W^a_t + 2e^{-2H}\Sigma_{IJ}(\partial_r\alpha^I_t)B^J\epsilon_{ab}W^b_t \\
= \; & -2kgZ\partial_r\beta^a - 4e^{-2H}\Sigma_{IJ}\partial_r\alpha^I_t\partial_t\alpha^J_a - 2ge^{-2H}\Sigma_{IJ}B^J\epsilon_{ab}\partial_r\alpha^I_b\,, \\
& \partial_r\{\Sigma_{IJ}e^H g^{-1/2}[(\partial_r\alpha^J_t)W^a_t + g\partial_r\alpha^J_a]\} = g^{-1/2}e^{-H}\Sigma_{IJ}(\partial_t^2\alpha^J_a + \frac{1}{2}\epsilon_{ab}\partial_t W^b_t B^J, \\
& \partial_r^2\beta^a + (3\partial_r H + \frac{\partial_r g}{2g} + \frac{\partial_r Z}{Z})\partial_r\beta^a - \frac{e^{-2H}}{g}\partial_r^2\beta^a = -g^{-1}e^{-2H}k\partial_t W^a_t.
\end{aligned} \tag{6.3.3}$$

As mentioned above, equations for W^t_a follows from the above set of equations. We further assume the time dependence of the various functions are $e^{i\omega t}$. This reduces above set of equations to the following

$$
\begin{aligned}
& [\partial_r^2 + (3\partial_r H - \frac{\partial_r g}{2g})\partial_r - e^{-2H}(2k^2 Z + e^{-2H}\Sigma_{IJ}B^I B^J)]W_t^a \\
= \; & -2e^{-2H}[-i\omega k Z\beta^a + 2\Sigma_{IJ}(\partial_r \alpha_t^I)(\partial_r \alpha_a^J) + i\omega e^{-2H}\Sigma_{IJ}\alpha_b^I \epsilon_{ab} B^J], \\
& i\omega\partial_r W_t^a + 2e^{-2H}\Sigma_{IJ}(\partial_r \alpha_t^I)B^J \epsilon_{ab} W_t^b \\
= \; & -2kgZ\partial_r \beta^a - 4i\omega e^{-2H}\Sigma_{IJ}\partial_r \alpha_t^I \alpha_a^J - 2ge^{-2H}\Sigma_{IJ}B^J \epsilon_{ab}\partial_r \alpha_b^I, \\
& \partial_r\{\Sigma_{IJ}e^H g^{-1/2}[(\partial_r \alpha_t^J)W_t^a + g\partial_r \alpha_a^J]\} = g^{-1/2}e^{-H}\Sigma_{IJ}(-\omega^2 \alpha_a^J + \frac{i\omega}{2}\epsilon_{ab}W_t^b B^J), \\
& \partial_r[e^{3H}g^{1/2}Z\partial_r \beta^a] = -i\omega k Z e^H g^{-1/2}(W_t^a - \frac{i\omega}{k}\beta^a).
\end{aligned}
\tag{6.3.4}
$$

we follow [88], to introduce new field

$$
\Theta^a = W_t^a - \frac{i\omega}{k}\beta^a. \tag{6.3.5}
$$

The fluctuation Θ^a is dual to the energy operator in the boundary theory. We introduce new function $\Omega = \omega^2 - 2k^2 gZ$ in order to express the equations in terms of this field Θ^a. Some of the terms, however, we have expressed in terms of W_t^a, which can be written in terms of Θ^a and β^a.

$$
\begin{aligned}
& \partial_r[2k^2 gZ\Omega^{-1}(-g^{-1/2}e^{3H}\partial_r\Theta^a + 4q_I\alpha_a^I) - 2i\omega\Omega^{-1}B^I\epsilon_{ab}(q_I W_t^b \\
- \; & g^{1/2}e^H\Sigma_{IJ}\partial_r\alpha_b^J)] - e^H g^{-1/2}(2k^2 Z + e^{-2H}\Sigma_{IJ}B^I B^J)\Theta^a \\
= \; & \frac{i\omega}{k}e^{-H}g^{-1/2}\Sigma_{IJ}B^I B^J\beta^a - 2i\omega e^{-H}g^{-1/2}\epsilon_{ab}\Sigma_{IJ}B^J\alpha_b^I, \\
- \; & g^{-1/2}e^H\partial_r(g^{1/2}e^H\Sigma_{IJ}\partial_r\alpha_a^J - 2k^2 gZ\Omega^{-1}q_I\Theta^a) - \frac{2k^2\omega^2}{\Omega^2}g^{-1/2}e^H\partial_r(gZ)q_I\Theta^a \\
+ \; & \omega^2 g^{-1}(\Sigma_{IJ} - 4\Omega^{-1}e^{-2H}gq_Iq_J)\alpha_a^J \\
+ \; & 2i\omega\Omega^{-1}e^{-2H}\epsilon_{ab}q_I B^J(q_J W_t^b - e^H g^{1/2}\Sigma_{JK}\partial_r\alpha_b^K) - \frac{i\omega}{2}g^{-1}\Sigma_{IJ}\epsilon_{ab}W_t^b B^J = 0, \\
& \partial_r W_t^a + 4e^{-2H}\Sigma_{IJ}(\partial_r\alpha_t^I)\alpha_a^J = \frac{2i}{\omega}e^{-2H}\Sigma_{IJ}B^J\epsilon_{ab}(\partial_r\alpha_t^J W_t^b + g\partial_r\alpha_b^J) + \frac{2ikgZ}{\omega}\partial_r\beta^a, \\
& \partial_r[e^{3H}g^{1/2}Z\partial_r\beta^a] = -i\omega kZe^H g^{-1/2}\Theta^a.
\end{aligned}
\tag{6.3.6}
$$

To analyse the theory at near horizon limit, we introduce another radial coordinate u, convenient for this purpose. it is related to r through $du = -g(r)^{1/2}e^{-H(r)}dr$. In terms of u the metric turns out to be

$$ds^2 = e^{2H(u)}(-g(u)dt^2 + \frac{du^2}{g(u)} + dxdx + dydy). \tag{6.3.7}$$

The derivatives with respect to u and r are through

$$\partial_r = -\sqrt{g}e^{-H}\partial_u, \quad \partial_u = -g^{-1/2}e^H\partial_r \tag{6.3.8}$$

The differential of u is expressed in terms of that of z through the relation $du = sgn(\xi)z^{v-3}dz$ where v and ξ are parameters determining behavior of the metric. The horizon of the black hole is located at $u = u_h$, so that the blackening factor in terms of u at that point vanishes, $g(u_h) = 0$ and at near horizon $g(r) \equiv 4\pi T\rho + \mathcal{O}(\rho^2)$, where $\rho = u_h - u$. At the near horizon limit, other quantities, A, Z and Σ_{IJ} approaches constant values.

The near horizon limit of the four equations are as follows:.

$$\begin{aligned}
& \frac{2k^2Z}{\omega^2}[g\partial_u(g\partial_u(e^{2H}\Theta^a))] + 2k^2Ze^{2H}\Theta^a - \frac{2i}{\omega}\epsilon_{ab}\Sigma_{IJ}B^I[g\partial_u(g\partial_u\alpha_b^J) + \omega^2\alpha_b^J] \\
+ & \frac{8k^2Z}{\omega^2}q_Ig\partial_u(g\alpha_a^I) - \frac{2i}{\omega}q_IB^I\epsilon_{ab}\partial_uW_t^b + \Sigma_{IJ}B^IB^JW_t^a = 0, \\
& \Sigma_{IJ}[g\partial_u(g\partial_u\alpha_a^J) + \omega^2\alpha_a^J] - \frac{2k^2Zq_I}{\omega^2}g^2\partial_u\Theta^a - 4e^{-2H}q_Iq_Jg\alpha_a^J \\
+ & \frac{2i}{\omega}e^{-2H}q_IB^J\epsilon_{ab}\Sigma_{JK}g^2\partial_u\alpha_b^K + \frac{2i}{\omega}e^{-2H}\epsilon_{ab}q_Iq_JB^JgW_t^b - \frac{i\omega}{2}\Sigma_{IJ}\epsilon_{ab}W_t^bB^J = 0, \\
& \partial_uW_t^a - \frac{2i}{\omega}e^{-2H}\epsilon_{ab}q_IB^IW_t^b + 4e^{-2H}q_I\alpha_a^I - \frac{2i}{\omega}e^{-2H}\epsilon_{ab}\Sigma_{IJ}B^Jg\partial_u\alpha_b^J \\
- & \frac{2ik}{\omega}g\partial_u\beta^a = 0, \\
& g\partial_u(gZ\partial_ue^{2H}\beta^a) = -i\omega kZe^{2H}\Theta^a.
\end{aligned} \tag{6.3.9}$$

Considering the terms contributing in leading order of ρ we obtain

$$\begin{aligned}
& \frac{2k^2 Z}{\omega^2}[g\partial_u(g\partial_u(e^{2H}\Theta^a))] + 2k^2 Z e^{2H}\Theta^a - \frac{2i}{\omega}\epsilon_{ab}\Sigma_{IJ}B^I[g\partial_u(g\partial_u\alpha_b^J) + \omega^2\alpha_b^J] \\
+ & \ \Sigma_{IJ}B^I B^J W_t^a = 0, \\
& \Sigma_{IJ}[g\partial_u(g\partial_u\alpha_a^J) + \omega^2\alpha_a^J] - \frac{i\omega}{2}\Sigma_{IJ}\epsilon_{ab}W_t^b B^J = 0. \qquad (6.3.10)
\end{aligned}$$

Choosing the in-falling behaviour of the fields at horizon, we obtain the following near horizon behaviour

$$e^{2H}\Theta^a \sim \rho^{\frac{-i\omega}{4\pi T}}, \qquad \eta_a^I \sim \rho^{\frac{-i\omega}{4\pi T}}. \qquad (6.3.11)$$

where we have introduced the combination

$$\eta_a^I = \alpha_a^I + \frac{1}{2k}e^{2H}B^I\epsilon_{ab}\beta^b. \qquad (6.3.12)$$

As we will see the above near horizon behaviour determines the relations among the integration constants appearing in the solutions of the different fields.

In order to study direct conductivity, we need to have solution of the fields Θ^a, α_a^I and β^a. The differential equations are quite complicated to solve but we do not require the full solution. Since direct conductivity depends on the behaviour of the fields at low frequency limit only, it is sufficient for our purpose to expand the fields in powers of frequency and determine the low frequency behaviour of the fields. In other words, we consider the following power series expansion

$$\begin{aligned}
\Theta^a &= \Theta^{a(0)} + \omega\Theta^{a(1)} + \omega^2\Theta^{a(2)} + ..., \\
\alpha_a^I &= \alpha_a^{I(0)} + \omega\alpha_a^{I(1)} + \omega^2\alpha_a^{(2)} + ..., \\
\beta^a &= \beta^{a(0)} + \omega\beta^{a(1)} + \omega^2\beta^{a(2)} + ... \qquad (6.3.13)
\end{aligned}$$

These expansions will enable us to determine the fields at different orders of frequency from the equations in an iterative manner .

We can separate out the equations to every order in frequency where first we consider the equation with the terms at the order of zero frequency. On substitution of the expansions of (6.3.13) in (6.3.6) yields from the second equation in (6.3.6)

$$\partial_r(g^{1/2}e^H\Sigma_{IJ}\partial_r\alpha_a^{J(0)} - q_I\Theta^{a(0)}) = 0. \tag{6.3.14}$$

This suggests we can introduce a new function

$$C_I^a = g^{1/2}e^H\Sigma_{IJ}\partial_r\alpha_a^J - q_I\Theta^a. \tag{6.3.15}$$

Then from (6.3.14) we understand that $C_I^{a(0)}$ is a constant. From the first equation in (6.3.6) we obtain

$$\partial_r[e^{3H}g^{3/2}\partial_r(g^{-1}\Theta^{a(0)}) + 4\alpha_t^I C_I^{a(0)}] = 0, \tag{6.3.16}$$

where we have used (6.2.6). For fluctuation of axion field we get from the third equation of (6.3.6)

$$\partial_u\beta^{a(0)} = \epsilon_{ab}\frac{C_I^{b(0)}B^I}{e^{2H}Z}g^{-1}. \tag{6.3.17}$$

From (6.3.14) and (6.3.16) we write the solutions in terms of the following integrals

$$\begin{aligned}
\Theta^{a(0)} &= g\Theta_1^a + g\Theta_2^a\int\frac{du}{e^{2H}g^2} - 4gC_I^{a(0)}\int\frac{\alpha_t^I du}{e^{2H}g^2}, \\
\alpha_a^{I(0)} &= \alpha_{a0}^{I(0)} - C_I^{a(0)}\int\frac{\Sigma^{IJ}}{g}du - q_J\Theta_1^a\int\Sigma^{IJ}du - q_J\Theta_2^a\int\Sigma^{IJ}\int\frac{du}{e^{2H}g^2} \\
&\quad - 4q_JC_K^{a(0)}\int du\Sigma^{IJ}\int\frac{\alpha_t^K du}{e^{2H}g^2}, \\
\beta^{a(0)} &= \beta_0^{a(0)} + \epsilon_{ab}C_I^{b(0)}B^I\int\frac{du}{e^{2H}gZ},
\end{aligned} \tag{6.3.18}$$

where we have used integration constants, Θ_1^a, Θ_2^a, $\alpha_{a0}^{I(0)}$ and $\beta_0^{a(0)}$.

At the near horizon limit, H, Z and Σ_{IJ} are taking constant value given by $H(h)$,

$Z(h)$ and $\Sigma_{IJ}(h)$. Behaviour of $g(u)$ near $u \to u_h$ is $g \sim 4\pi T\rho$ and $\alpha_t^I \sim \mathcal{O}(\rho)$, which gives the expression

$$\begin{aligned}
\Theta^{a(0)} &= (4\pi T\rho)\Theta_1 + \frac{\Theta_2^a}{4\pi T e^{2H(h)}} - \frac{4C_I^{a(0)}\partial_u\alpha_t^I}{4\pi T e^{2H(h)}}\rho\log\rho, \\
\alpha_a^{I(0)} &= \alpha_{a0}^{I(0)} + (\frac{q_J\Theta_2^a}{4\pi T e^{2H(h)}} + C_J^{a(0)})\frac{\Sigma^{IJ}(h)}{4\pi T}\log\rho + q_J\Theta_1^a\Sigma^{IJ}(h)\rho, \\
\beta^{a(0)} &= \beta_0^{a(0)} - \epsilon_{ab}\frac{C_I^{b(0)}B^I}{4\pi T e^{2H(h)}Z(h)}\log\rho.
\end{aligned} \tag{6.3.19}$$

Since B^I enters the equations at the first order of ω, The equations are very much similar to the one obtained in absence of magnetic field [88] at the zeroeth order of frequency.

Next we will consider the equations with the terms at first order of frequency. Here we will adopt a recursive procedure to determine the solutions at different orders of ω, by using solutions obtained in the lower orders. On substitution of (6.3.13) in the second equation in (6.3.6) we get

$$\begin{aligned}
& g^{-1/2}e^H\partial_r(g^{1/2}e^H\Sigma_{IJ}\partial_r\alpha_a^{J(1)} - q_I\Theta^{a(1)}) \\
- \ & \frac{2i}{2k^2gZ}e^{-2H}\epsilon_{ab}q_IB^J(q_J\Theta^{b(0)} - e^Hg^{1/2}\Sigma_{JK}\partial_r\alpha_b^{K(0)}) \\
- \ & \frac{i}{2}g^{-1}\Sigma_{IJ}\epsilon_{ab}\Theta^{b(0)}B^J = 0,
\end{aligned} \tag{6.3.20}$$

which leads to

$$\partial_u C_I^{a(1)} = -ig^{-1}\epsilon_{ab}B^J(-\frac{q_IC_J^{b(0)}}{p^2Ze^{2H}} + \Sigma_{IJ}\frac{\Theta^{b(0)}}{2}). \tag{6.3.21}$$

By integrating (6.3.21) we can write $C_I^{a(1)}$ in terms of the zeroeth order terms. Similarly, $\Theta^{a(1)}$ and $\beta^{a(1)}$ satisfy the relation

$$\begin{aligned}
& \partial_u[e^{2H}g^2\partial_u(g^{-1}\Theta^{a(1)})] - 4C_I^{a(1)}\partial_u\alpha_t^I + 2i\epsilon_{ab}\Sigma_{IJ}B^J\alpha_b^{I(0)} \\
& + \frac{i}{k^2}\epsilon_{ab}B^IgC_I^{a(0)}\partial_u(\frac{1}{gZ}) - \frac{i}{k}\Sigma_{IJ}B^IB^J\beta^{a(0)} = 0, \\
& \partial_u[e^{2H}HZ\partial_u\beta^{a(1)}] = ikZe^{2H}g^{-1}\Theta^{a(0)},
\end{aligned} \tag{6.3.22}$$

while we can get $\alpha_a^{I(1)}$ from

$$\partial_u \alpha_a^{I(1)} = -g^{-1}\Sigma^{IJ} C_J^{a(1)} - q_I g^{-1}\Theta^{a(1)}. \tag{6.3.23}$$

Like $C_I^{a(1)}$, we can integrate all these equations to obtain expressions at first order in terms of the zeroeth order fields.

Using the recursive procedure as we mentioned, we substitute the near horizon behaviour of g, H, Z and Σ obtained for the zeroeth order and integrate the above equations to get the near horizon behaviour of the fields at first order. The expression for $C_I^{a(1)}$ is

$$\begin{aligned} C_I^{a(1)} =& C_{I0}^{a(1)} + \frac{i\epsilon_{ab}B^J}{e^{2H(h)}4\pi T}[\left(-\frac{q_I C_J^{b(0)}}{k^2 Z(h)} + \frac{\Theta_2^b \Sigma_{IJ}(h)}{8\pi T}\right)\log\rho + \frac{1}{2}\Theta_1^a \Sigma_{IJ}(h) e^{2H(h)} 4\pi T\rho \\ &+ \frac{2\Sigma_{IJ} C_K^{a(0)} \partial_u \alpha_t^K}{4\pi T}(\rho\log\rho - \rho)] + ..., \end{aligned} \tag{6.3.24}$$

where $C_{I0}^{a(1)}$ is the constant of integration.

From this expressions, we can obtain a similar near horizon expression for $\Theta^{a(1)}$ from (6.3.22) as follows

$$\Theta^{a(1)} = \frac{\Theta_3^a}{e^{2H(h)}4\pi T} + \frac{i}{k^2}\frac{\epsilon_{ab}B^I C_I^{a(0)}}{Z(h)}\log\rho + \Theta_4^a 4\pi T\rho + ..., \tag{6.3.25}$$

where Θ_3^a and Θ_4^a are constants of integration. The fluctuation in gauge field at first order, $\alpha_a^{I(1)}$ at the near horizon limit can be obtained from (6.3.23) and is given by

$$\alpha_a^{I(1)} = \alpha_{a0}^{I(1)} + \frac{\Sigma^{IJ}(h)}{4\pi T}\left[\frac{q_J \Theta_3^a}{e^{2H(h)}4\pi T} + C_{J0}^{a(1)}\right]\log\rho + ..., \tag{6.3.26}$$

where we have introduced the integration constant as $\alpha_{a0}^{I(1)}$. Finally the β^a at first order appeared to be

$$\begin{aligned}
\beta^{a(1)} &= \beta_0^{a(1)} \\
&- \frac{e^{-2H(h)}}{4\pi TkZ(h)}[\epsilon_{ab}B^K C_{K0}^{b(1)} - \frac{i}{2}(e^{2H}4\pi T\Theta_1^a + 4q_i\alpha_{a0}^{I(0)}) \\
&+ \frac{i}{k}q_I B^I \epsilon_{ab}\beta_0^{b(0)}]\log\rho + ...,
\end{aligned} \tag{6.3.27}$$

We can obtain the integration constants, which we introduced at different orders in the following manner. We compare the expressions near the horizon behaviour with the near horizon behaviour of the various fluctuations, as given in (6.3.11). That requires consideration of the equations upto second order in frequency.

From the second order of ω equation we obtain the expression for $C_I^{a(2)}$ as

$$\begin{aligned}
\partial_u C_I^{a(2)} =& \frac{q_I e^{-2H}}{2k^2 gZ}[(e^{2H}\partial_u\Theta^{a(0)} + 4q_J\alpha_a^{J(0)}) + 2i\epsilon_{ab}(C_J^{b(1)} - \frac{i}{k}q_J\beta^{b(0)})B^J] \\
&- \frac{\Sigma_{IJ}}{g}[\alpha_a^{J(0)} - \frac{i}{2}(\Theta^{b(1)} + \frac{i}{k}\beta^{b(0)})B^J]
\end{aligned} \tag{6.3.28}$$

On the other hand for $\Theta^{a(2)}$ we get

$$\begin{aligned}
\partial_u[e^{2H}g^2\partial_u(g^{-1}\Theta^{a(2)})] =& 4C_I^{a(2)}\partial_u\alpha_t^I - g\partial_u[\frac{1}{2k^2 gZ}[(e^{2H}\partial_u\Theta^{a(0)} + 4q_J\alpha_a^{J(0)}) \\
&+2i\epsilon_{ab}(C_J^{b(1)} - \frac{i}{k}q_J\beta^{b(0)})B^J] + \frac{i}{k}\Sigma_{IJ}B^I B^J\beta^{a(1)} - 2i\epsilon_{ab}\alpha_b^{I(1)}.
\end{aligned} \tag{6.3.29}$$

$$C_I^{a(2)} = g^{1/2}e^H\Sigma_{IJ}\partial_r\alpha_a^{J(2)} - q_I\Theta^{a(2)}. \tag{6.3.30}$$

To compare to the boundary condition at horizon we need to obtain the leading order behaviour of the fields near the horizon. We substitute the zeroeth order and first order expressions of the fields on right hand side of (6.3.28) and observe that at the near horizon limit, the leading order expressions of $C_I^{a(2)}$ terms of the order of $\log\rho$ and $(\log\rho)^2$. In particular, there is no $1/\rho$ term in its near horizon leading order expression. From equation for $\Theta^{a(2)}$, we obtain the leading order behaviour of $\Theta^{a(2)}$ as

$$g^{-1}\Theta^{a(2)} = \Theta_6 + \Theta_5 \int \frac{du}{e^{2H}g^2} + S\log\rho + ... \tag{6.3.31}$$

We have introduced two constants of integration, Θ_5^a and Θ_6^a. The coefficient of $\log\rho$ written as S is

$$S = \frac{1}{2k^2 Z(h)}[(-4\pi T e^{2H}\Theta_1^a + 4q_I\alpha_{a0}^I) + 2i\epsilon_{ab}(C_{I0}^{b(1)} - \frac{i}{k}q_I\beta_0^{b(0)})B^I] + \frac{\Theta_2^a}{(4\pi T)^2}. \tag{6.3.32}$$

We collect the expressions of Θ^a at different orders of frequency together to write the near horizon expression of Θ^a upto $\mathcal{O}(\omega^2)$ order as

$$\begin{aligned}\Theta^a =& \frac{\Theta_2^a}{e^{2H(h)}4\pi T} + 4\pi T\Theta_1^a\rho + (\frac{1}{4\pi T e^{2H(h)}}[\frac{i\omega\epsilon_{ab}C_{I0}^b}{k^2 Z(h)} + \frac{2(-\pi T e^{2H(h)}\Theta_1^a + q_I\alpha_{a0}^{I0})}{k^2 Z(h)}\omega^2 \\ &+ \frac{\omega^2}{k^3 Z(h)}\epsilon_{ab}\beta_0^{b(0)}q_I B^I] + \omega^2\frac{\Theta_2^a}{(4\pi T)^2})\log\rho + ...\end{aligned} \tag{6.3.33}$$

Following [88] we redefine Θ_2^a, Θ_1^a and C_{I0}^a so as to absorbed all the constants of integration in them . Similarly the fluctuation of gauge field at near horizon limit is given by

$$\alpha_a^I = \alpha_{a0}^I + \frac{\Sigma^{IJ}(h)}{4\pi T}(C_{J0}^a + \frac{q_J\Theta_2^a}{e^{2H(h)}4\pi T})\log\rho + ... \tag{6.3.34}$$

where, once again, α_{a0}^I absorbs all the constants of integration. Fluctuation of the axion field β^a at near horizon limit given by

$$\begin{aligned}\frac{\beta^a}{k} &= \frac{\beta_0^a}{k} + \frac{1}{4\pi T e^{2H(h)}k^2 Z(h)}[-\epsilon_{ab}B^I C_{I0}^b + 2i\omega(-\pi T e^{2H}(h)\Theta_1^a + q_I\alpha_{0a}^I) \\ &+ i\omega(q_I B^I)\epsilon_{ab}\frac{\beta_0^{b(0)}}{k}]\log\rho + ...\end{aligned} \tag{6.3.35}$$

where the constants of integrations are absorbed in β_0^a.

Comparison with the near horizon behaviours of Θ^a and $\eta_a^I = \alpha_a^I + \frac{1}{2}B^I\epsilon_{ab}\frac{\beta^b}{k}$ as given in (6.3.11), leads to

$$(\Sigma^{IJ}(h) + \frac{B^I B^J}{2k^2 Z e^{2H(h)}})C^a_{J0} + \frac{\Sigma^{IJ}(h) q_J}{e^{2H(h)} 4\pi T}\Theta^a_2 = i\omega\{(\alpha^I_{a0} + \frac{1}{2}B^I \epsilon_{ab}\frac{\beta^b_0}{k}) - \frac{B^I}{e^{2H(h)}k^2 Z}[\epsilon_{ab} (-\pi T e^{2H(h)}\Theta^b_1 + q_I \alpha^I_{0b}) + \frac{1}{2}(q_J B^J)\frac{\beta^a_0}{k}\},$$
$$\Theta^a_2 = -\frac{4\pi T}{k^2 Z(h)}[\epsilon_{ab} B^I C^b_{I0} - 2i\omega(-\pi T e^{2H(h)}\Theta^a_1 + q_I \alpha^I_{a0}) - i\omega(q_J B^J)\epsilon_{ab}\frac{\beta^b_0}{k}]. \tag{6.3.36}$$

From the above two equations (6.3.36), C^a_{I0} and Θ^a_2 can be expressed in terms of other constants α^I_{a0}, Θ^a_1 and β^a_0 in the following way,

$$\begin{aligned} C^a_{I0} &= i\omega(M_I{}^J)_{ab}\{-[(\Sigma_{JK}(h) + \frac{2q_J q_K}{k^2 Z(h) e^{2H(h)}})\delta_{bc} + \frac{\Sigma_{JN}(h) B^N q_K}{k^2 Z(h) e^{2H(h)}}\epsilon_{bc}]\alpha^k_{c0} \\ &+ \frac{2\pi T}{k^2 Z(h)}[q_J \delta_{ab} + \frac{1}{2}\Sigma_{JN}(h) B^N \epsilon_{bc}]\Theta^c_1 - \frac{1}{2}[(\Sigma_{JK}(h) \\ &+ \frac{4 q_J q_K}{k^2 Z(h) e^{2H(h)}})B^K \epsilon_{bc} - (q_M B^M)\frac{\Sigma_{JK}(h) B^K}{k^2 Z(h) e^{2H(h)}}\delta_{bc}]\frac{\beta^c_0}{k}\}, \\ \Theta^a_2 &= -\frac{4\pi T}{k^2 Z(h)}\epsilon_{ab} B^I C^b_{I0} + i\omega\frac{4\pi T}{k^2 Z(h)}[2(-\pi T e^{2H(h)}\Theta^a_1 + q_I \alpha^I_{0a}) + (q_I B^I)\epsilon_{ab}\frac{\beta^b_0}{k}], \end{aligned} \tag{6.3.37}$$

upto leading order in ω. In the above equation we have introduced the matrix $(M^J_I)_{ab}$ given by the relation

$$[(\delta^J_I + \frac{\Sigma_{IN}(h) B^N B^J}{2k^2 Z(h) e^{2H(h)}})\delta_{ab} - \frac{q_I B^J}{k^2 Z(h) e^{2H(h)}}\epsilon_{ab}](M_J{}^K)_{bc} = \delta^K_I \delta_{ac}. \tag{6.3.38}$$

It reduces to $\delta^J_I \delta_{ab}$ in the absence of magnetic field.

To identify the right operators in the boundary theory, we need the asymptotic solution of Θ^a, α^I_a and β^a and the asymptotic solution of the fields upto the lowest order in frequency will be sufficient for this purpose. The magnetic field contributes at a higher order in frequency, as evident from the linearised equations of motion of the fluctutations. Therefore, upto the lowest order of frequency, the expressions is same as those obtained in [88] for zero magnetic field . Hence we introduce

$$\begin{aligned}
\Psi(z) &= sgn(\theta)\int \frac{z^{\xi-3v-1}dz}{F(z)^2},\\
Y^1(z) &= \frac{4sgn(\xi)q_1}{2+v-\xi}(-z_h^{2+v-\xi}\Psi(z) + sgn(\xi)\int dz z^{-2v+1}F^{-2}),\\
Y^2(z) &= \frac{4sgn(\xi)q_2}{\xi-z}(-z_h^{\xi-v}\Psi(z) + sgn(\xi)\int dz z^{2\xi-4v-1}F^{-2}).
\end{aligned} \tag{6.3.39}$$

In terms of the functions defined above we can write down the asymptotic expansions of the solutions of the fields at small frequency

$$\begin{aligned}
\Theta^{a(0)} &= z^{2(v-1)}F(z)(\Theta_1^a + \Theta_2^a\Psi(z) + 4C_I^aY^I(z)),\\
\alpha_a^I &= \alpha_{a0}^I - \Theta_1^a\alpha_t^I - sgn(\xi)\Theta_2 q_J\int dz\Sigma^{IJ}z^{v-3}\Psi(z)\\
&- sgn(\xi)\int dz\Sigma^{IJ}z^{-v-1}(F^{-1}\delta_K^J + 4q_J z^{2(v-1)}Y^K(z))C_K^a.
\end{aligned} \tag{6.3.40}$$

From (6.3.40) we can establish a relation between the parameters describing the asymptotic behaviour of the solutions and operators in the boundary theory. A discussion of this relation is given in [88], which we have included in the Appendix B. As explained in [88], one can identify the basis of symplectic variables which parametrizes the asymptotic solutions from asymptotic behaviour of the generalised coordinates and momenta. For that one expresses asymptotic solutions of the linear fluctuations of the fields $\Theta^{a(0)}$, $\alpha_a^{I(0)}$ and $\beta^{a(0)}$ and their conjugate momenta in terms of the modes Θ_1^a, Θ_2^a, α_{a0}^I, C_I^a and β_0^a in the radial Hamiltonian formalism. Next one makes a suitable canonical transformation, which can be realised by adding appropriate counterterms, leads to holographic renormalisation of the action. As we see that from the asymptotic behaviour of these transformed canonical variables, we can identify the operators in terms of the modes parametrizing the asymptotic solution.

The choice of the boundary condition appears to play an important role in this identification. As explained in [88], the Dirichlet boundary condition can be imposed on the gauge field through addition of a finite term in the renormalised on-shell action. We see that with the electrically charged black hole as the background, the expressions of the conductivities obtained using the near horizon method agrees with the Dirichlet boundary condition. Therefore, in order to compare the results already obtained using near horizon method, we will consider the Dirichlet boundary condition in the present case, where we have magenteic field in the background. Generalisation of these to other boundary conditions like Neumann or mixed boundary condition is quite straightforward.

In the case of the Dirichlet boundary condition, one is interested in energy operator $\mathcal{E}^a$ and current operator $\mathcal{J}_I^a$ as shown in [88]. The expressions of these operators in terms of different modes are given by (B.0.10) and (B.0.11)

$$\mathcal{E}^a = -\frac{1}{2\kappa^2}(\Theta_2^a + 4\mu^I C_{I0}^a), \quad \mathcal{J}_I^a = -\frac{2}{\kappa^2}(C_{I0}^a - \frac{i\omega q_I}{p}\beta_0^a), \quad \mathcal{X}_a = -\frac{2i\omega}{p\kappa^2}q_I\alpha_a^I. \tag{6.3.41}$$

where α_a^I is obtained from the asymptotic behaviour for the renormalised variables as given in (B.0.12). Various correlation functions can be obtained from the expressions of operators,which has been used for computation of the coefficients of thermoelectric conductivity.

6.4 Thermoelectric DC conductivities

In this section we discuss thermoelectric conductivities for the model which we are considering. We have determined Θ_2^a and C_{I0}^a in terms of other constants (6.3.36) in the last section. Substituting these expressions in the energy operator $\mathcal{E}^a$ given in

(6.3.41), we obtain

$$
\begin{aligned}
\mathcal{E}^a &= -\frac{i\omega}{2\kappa^2}[\{\frac{8\pi T}{k^2 Z}q_K\delta_{ad} - (-\frac{4\pi T}{k^2 Z}\epsilon_{ab}B^I + 4\mu^I\delta_{ab})(M_I{}^J)_{bc}[(\Sigma_{JK} + \frac{4q_Jq_K}{k^2 Ze^{2A}})\delta_{cd} \\
&+ \frac{\Sigma_{JM}B^M q_k}{k^2 Ze^{2A}}\epsilon_{cd}\}\alpha^K_{d0} + (\frac{8\pi T}{k^2 Z}(q_K\mu^K - \pi Te^{2H})\delta_{ad} \\
&+ (-\frac{4\pi T}{k^2 Z}\epsilon_{ab}B^I + 4\mu^I\delta_{ab})(M_I^J)_{bc}\{\frac{2\pi T}{2k^2 Z}(q_J\delta_{cd} + \frac{1}{2}\Sigma_{JM}B^M\epsilon_{cd}) \\
&- [(\Sigma_{JK} + \frac{2q_Jq_K}{k^2 Ze^{2A}})\mu^K\delta_{cd} + \frac{\Sigma_{JM}B^M q_K\mu^K}{k^2 Ze^{2A}}\epsilon_{cd}]\})\Theta^d_1 \\
&+ \{-\frac{1}{2}(-\frac{4\pi T}{k^2 Z}\epsilon_{ab}B^I + 4\mu^I\delta_{ab})(M_I{}^J)_{bc}[(\Sigma_{JK} + \frac{4q_Jq_K}{k^2 Ze^{2A}})B^K\epsilon_{cd} \\
&- \frac{(q_KB^K)\Sigma_{JM}B^M}{k^2 Ze^{2A}}\delta_{cd}] + \frac{4\pi T}{k^2 Z}(q_KB^K)\epsilon_{ad}\}\frac{\tau^d_0}{k}.
\end{aligned}
\tag{6.4.1}
$$

In the above equations, we have used the asymptotic value of fluctuation in gauge field, α^I_a as given in (B.0.12). In this section, to simplify the notation, we follow the convention that, unless otherwise mentioned, A, Σ_{IJ} and Z represents their respective values at the near horizon limit.

The current operator $\mathcal{J}^a_I$ is given by

$$
\begin{aligned}
&J^a_I = \frac{2i\omega}{\kappa^2}[(M_I{}^J)_{ab}[(\Sigma_{JK} + \frac{2q_Jq_K}{k^2 Ze^{2A}})\delta_{bc} + \frac{\Sigma_{JM}B^M q_K}{k^2 Ze^{2A}}\epsilon_{bc}]\alpha^K_{c0} \\
&- (M_I{}^J)_{ab}\{\frac{2\pi T}{2k^2 Z}(q_J\delta_{bc} + \frac{1}{2}\Sigma_{JM}B^M\epsilon_{bc}) \\
&- [(\Sigma_{JK} + \frac{2q_Jq_K}{k^2 Ze^{2H}})\mu^K\delta_{bc} + \frac{\Sigma_{JM}B^M q_K\mu^K}{k^2 Ze^{2H}}\epsilon_{bc}]\}\Theta^c_1 \\
&+ \{\frac{1}{2}(M_I{}^J)_{ab}[(\Sigma_{JK} + \frac{2q_Jq_K}{k^2 Ze^{2A}})B^K\epsilon_{bc} - (q_NB^N)\frac{\Sigma_{JK}B^K}{k^2 Ze^{2A}}\delta_{bc}] + q_I\delta_{ac}\}\frac{\tau^c_0}{k}, \\
&\mathcal{X}_a = -\frac{2i\omega}{k\kappa^2}q_I\alpha^I_a,
\end{aligned}
\tag{6.4.2}
$$

where we have taken the expression of matrix $(M_I^J)_{ab}$ from (6.3.38).

The above equations lead to the expressions of two-point functions as follows

$$\begin{aligned}
\langle \mathcal{J}_I^a(-\omega)\mathcal{J}_J^b(\omega)\rangle &= \frac{2i\omega}{\kappa^2}(M_J{}^K)_{bc}[(\Sigma_{KI}+\frac{2q_Kq_I}{k^2Ze^{2A}})\delta_{ca}+\frac{\Sigma_{KM}B^Mq_I}{k^Ze^{2H}}\epsilon_{ca}],\\
\langle \mathcal{E}^a(-\omega)\mathcal{J}_I^b(\omega)\rangle &= -\frac{2i\omega}{\kappa^2}(M_I^J)_{bc}\{\frac{2\pi T}{k^2Z}(q_J\delta_{ca}+\frac{1}{2}\Sigma_{JK}B^K\epsilon_{ca})\\
&- [(\Sigma_{JK}+\frac{2q_Jq_K}{k^2Ze^{2A}})\delta_{bc}+\frac{\Sigma_{JM}B^Mq_K}{k^2Ze^{2A}}\epsilon_{bc}]\mu^K\\
\}, \langle \mathcal{J}_I^a(-\omega)\mathcal{E}^b(\omega)\rangle &= \frac{2i\omega}{\kappa^2}[(-\frac{2\pi T}{k^2Z}q_I\delta_{ba}-(-\frac{\pi T}{k^2Z}B^J\epsilon_{bc}+\mu^J\delta_{bc})(M_J{}^K)_{cd}\\
&\quad [(\Sigma_{KI}+\frac{2q_Kq_I}{k^2Ze^{2H}})\delta_{da}+\frac{\Sigma_{KM}B^Mq_I}{k^2Ze^{2H}}\epsilon_{da}]\},\\
\langle \mathcal{E}^a(-\omega)\mathcal{E}^b(\omega)\rangle &= \frac{2i\omega}{\kappa^2}[\frac{2\pi T}{k^2Z}(q_K\mu^K-\pi Te^{2H})\delta_{ba}+(-\frac{\pi T}{k^2Ze^{2H}}\epsilon^{bc}B^I+4\mu^I\delta_{bc})\\
&\quad (M_I^J)_{cd}[(\frac{2\pi T}{k^2Z}q_J-(\Sigma_{JK}+\frac{2q_Jq_K}{k^2Ze^{2H}})\mu^K]\delta_{da}\\
&+ \Sigma_{JM}B^M(\frac{\pi T}{k^2Z}-\frac{q_K\mu^K}{k^2Ze^{2H}})\epsilon_{da}],\\
\langle \mathcal{X}^a(-\omega)\mathcal{J}_I^b(\omega)\rangle &= \frac{2i\omega}{\kappa^2}[\frac{1}{2}(M_I^J)_{bc}[(\Sigma_{JK}+\frac{4q_Jq_K}{k^2Ze^{2H}})B^K\epsilon_{ca}\\
&- (q_JB^J)\frac{\Sigma_{JK}B^K}{k^2Ze^{2H}}\delta^{ca}]+q_I\delta_{ba},\\
\langle \mathcal{J}_I^a(-\omega)\mathcal{X}^b(\omega)\rangle &= -\frac{2i\omega}{k\kappa^2}q_I\delta^{ab},
\end{aligned} \tag{6.4.3}$$

where all the other two point functions vanishes. We introduce the heat current following [88]

$$\mathcal{Q}_D^a = \mathcal{E}^a - \mu^I\mathcal{J}_I^a. \tag{6.4.4}$$

We obtain the expression for two point function for heat current and electric currents as follows

$$
\begin{aligned}
\langle \mathcal{Q}_D^a(-\omega)\mathcal{Q}_D^b(\omega)\rangle &= \frac{2i\omega}{\kappa^2}2(\frac{\pi T}{k^2 Z})^2\{k^2 Z e^{2H}\delta^{ab} \\
&+ B^M\epsilon_{ac}(M_M^{\ J})_{cd}[q_J\delta_{da}+\frac{1}{2}\Sigma_{JN}B^N\epsilon_{da}]\}, \\
\langle \mathcal{Q}_D^a(-\omega)\mathcal{J}_I^b(\omega)\rangle &= -\frac{2i\omega}{\kappa^2}\frac{2\pi T}{k^2 Z}[(M_I^{\ J})_{bc}[q_J\delta_{ca}+\frac{1}{2}\Sigma_{JK}B^K\epsilon_{ca}], \\
\langle \mathcal{J}_I^a(-\omega)\mathcal{Q}_D^b(\omega)\rangle &= -\frac{2i\omega}{\kappa^2}\{\frac{2\pi T}{k^2 Z}q_I\delta_{ba}+\frac{\pi T}{k^2 Z}\epsilon_{bc}B^J(M_J^{\ K})^{cd}[(\Sigma_{KI}+\frac{2q_K q_I}{k^2 Z e^{2H}})\delta_{da} \\
&+ \frac{\Sigma_{KM}B^M q_J}{k^2 Z E^2 A}\epsilon_{da}]\}, \\
\langle \mathcal{J}_I^a(-\omega)\mathcal{J}_J^b(\omega)\rangle &= \frac{2i\omega}{\kappa^2}(M_J^{\ K})_{bc}[(\Sigma_{KI}+\frac{2q_K q_I}{k^2 Z e^{2H}})\delta_{ca}+\frac{\Sigma_{KM}B^M q_I}{k^2 Z e^{2H}}\epsilon_{ca}]\,. \quad (6.4.5)
\end{aligned}
$$

From the expression of above two point functions, we compute the thermoelectric conductivities as follows.

$$
\sigma_D^{DC} = \begin{pmatrix} T\bar{\mathbb{K}}^{ab} & T\bar{a}_I^{ab} \\ Ta_I^{ab} & \nu_{IJ}^{ab} \end{pmatrix} = \begin{pmatrix} \langle \mathcal{Q}_D^a(-\omega)\mathcal{Q}_D^b(\omega)\rangle & \langle \mathcal{Q}_D^a(-\omega)\mathcal{J}_I^b(\omega)\rangle \\ \langle \mathcal{J}_I^a(-\omega)\mathcal{Q}_D^b(\omega)\rangle & \langle \mathcal{J}_I^a(-\omega)\mathcal{J}_J^b(\omega)\rangle \end{pmatrix}. \quad (6.4.6)
$$

We introduce the following parameters in order to express the components of conductivity matrix in a compact form

$$
n_I = \frac{1}{2}\Sigma_{IJ}B^J, \qquad b^I = \frac{B^I}{k^2 Z e^{2H}}. \quad (6.4.7)
$$

The matrix $(M_I^{\ J})_{ab}$ can be written from (6.3.38) using these parameters,

$$
(M_I^{\ J})_{ab} = \delta_I^J\delta_{ab} - \frac{[(1+n.b)n_I+(q.b)q_I]\delta_{ab}-[(1+n.b)q_I-(q.b)n_I]\epsilon_{ab}}{(1+n.b)^2+(q.b)^2}b^J. \quad (6.4.8)
$$

where we have defined $(n.b) = n_I b^I$, $(q.b) = q_I b^I$ and $\triangle = (1+n.b)^2+(q.b)^2$. With these expressions, one can write the components of conductivity matrix as

$$
\begin{aligned}
\bar{\mathbb{K}}^{ab} &= \frac{\pi s T}{\kappa^2 k^2 Z}\frac{[(1+n.b)\delta_{ba}+(q.b)\epsilon_{ba}]}{\triangle}, \\
\bar{a}_I^{ab} &= a_I^{ab} = -\frac{4}{sT}\bar{\mathbb{K}}^{bc}(q_I\delta_{ca}+n_I\epsilon_{ca})], \\
\nu_{IJ}^{ab} &= \frac{2}{\kappa^2}\Sigma_{JI}\delta^{ba}+\frac{16}{s^2 T}\bar{\mathbb{K}}^{bc}(q_J\delta_{cd}+n_J\epsilon_{cd})(q_I\delta_{da}+n_I\epsilon_{da}),
\end{aligned} \quad (6.4.9)
$$

where the entropy s is given by $s = 4\pi e^{2H}$. All the components of the conductivity matrix reduce to the expressions of the same given in [88] when we set the background magnetic field equal to zero. It may be noted that both the $U(1)$ gauge fields are on the same footing. We have obtained $\bar{a}_I^{ab} = a_I^{ab}$ implying time reversal symmetry.

The above equations summarise the expressions of coefficints of the thermoelectric conductivities for general case. One can observe that and these expressions are symmetric between and electric and magnetic fields. Next we consider dyonic black hole discussed in section 2 and apply this general result to that case. We substitute the values of the various quantities for dyonic black hole in the above expressions and obtain the following.

For the solution we obtain $\triangle = (k^2 + \frac{B^2}{4} z^{4v-6-\xi})^2 + (2q_2 B z^{2v-4})^2$ and using the relation we get,

$$\begin{aligned}
\bar{\mathbb{K}}^{mn} &= \frac{8\pi^2 T}{\kappa^2 k^2} z_h^{2(v-\xi)} \frac{(k^2 + \frac{B^2}{4} z_h^{4v-6-\xi})\delta_{mn} + 2q_2 B z_h^{2v-4}\epsilon_{mn}}{\triangle}, \\
a_1^{mn} &= -\frac{8\pi}{\kappa^2} z_h^{2v-\xi-2} \frac{(k^2 + \frac{B^2}{4} z_h^{4v-6-\xi}) q_1 \delta_{mn} + 2q_1 q_2 B z_h^{2v-4}\epsilon_{mn}}{\triangle}, \\
a_2^{mn} &= -\frac{8\pi}{\kappa^2} z_h^{2v-\xi-2} \frac{k^2 q_2 \delta_{mn} + [(k^2 + \frac{B^2}{4} v_h^{4z-6-\theta}) \frac{B}{8} z_h^{2v-2-\xi} + 2q_2^2 B z_h^{2v-4}]\epsilon_{mn}}{\triangle}, \\
\nu_{11}^{mn} &= \frac{1}{2\kappa^2} z_h^{\xi-4}\delta_{mn} + \frac{8}{\kappa^2} q_1^2 z_h^{2v-4} \frac{(k^2 + \frac{B^2}{4} z_h^{4v-6-\xi})\delta_{mn} + 2q_2 B z_h^{2v-4}\epsilon_{mn}}{\triangle}, \\
\nu_{12}^{mn} &= \frac{8}{\kappa^2} q_1 \frac{q_2 k^2 \delta_{mn} + [2q_2^2 B z_h^{2v-4} + \frac{B}{8} z_h^{2v-2-\xi}(k^2 + \frac{B^2}{4} z_h^{4v-6-\xi})]\epsilon_{mn}}{\triangle}, \qquad (6.4.10) \\
\nu_{22}^{mn} &= \frac{k^2}{2\kappa^2} z_h^{6v-8-2\xi} \frac{\frac{B^2}{4} + z_h^{6-4v+\xi}(k^2 + 16 q_2^2 z_h^{\xi-2})}{\triangle}\delta_{mn} \\
&+ \frac{q_2 B}{\kappa^2} z_h^{8v-12-2\xi} \frac{\frac{B^2}{4} + (2k^2 + 16 q_2^2 z_h^{\xi-2}) z_h^{-4v+6+\xi}}{\triangle}\epsilon_{mn}. \qquad (6.4.11)
\end{aligned}$$

Also one can obtain the expression of Hall angle from the above conductivities by taking the ratio of coefficients of ϵ_{ab} and δ_{ab} in the expression of σ. We obtain

$$\Theta_H = \frac{2q_2 B}{k^2} z_h^{2v-4} [\frac{\frac{B^2}{4} + z_h^{-4v+6+\xi}(2k^2 + 16 q_2^2 z_h^{\xi-2})}{\frac{B^2}{4} + z_h^{-4v+6+\xi}(k^2 + 16 q_2^2 z_h^{\xi-2})}]. \qquad (6.4.12)$$

One observes that the factor in the square bracket lies between 1 and 2 and therefore, Hall coefiicient can be approximated as [95]

$$\Theta_H = \frac{2q_2 B}{k^2} v_h^{2z-4} k^2. \tag{6.4.13}$$

The special case of the above expressions for $\theta = 1 - z$, is in agreement with the results given in [87], which they obtained the near horizon method, as expected.

Once we obtain the various components of thermoelectric matrix, we can study their temperature dependence. The temperature for the analytic black hole solution is given by $T = -\frac{sgn(\theta)}{4\pi} v_h^{z+1} F'(v_h)$. In the present case of dyonic solution it becomes

$$T = -\frac{sgn(\xi)}{4\pi}[(v+2-\xi)z_h^v - \frac{8q_2^2}{2-\xi} z_h^{2\xi-v-2} - \frac{k^2}{2-\xi} z_h^{\xi-z} - \frac{B^2}{4(2-v)} z_h^{3v-6}. \tag{6.4.14}$$

This expression of temperature makes it difficult to obtain an analytic expression of the conductivities as the series expansion in temperature. Nevertheless, while we choose the appropriate limits of the quantities we can identify different regimes, where one can discuss scaling behaviour of the coefficients with the temperature.

To begin with, we consider $\theta < 0$. For that condition the first term in the expression of temperature is positive while rest of the terms are negative . Then following [88] we consider the region of parameter space $q_2^2 z_h^{2\xi-v-2} << z_h^v$, $k^2 z_h^{\xi-v} << z_h^v$ and $B^2 z_h^{3(v-2)} << z_h^v$ to identify a regime of large temperature. In this region one can approximate the temperature by the expression as $T \equiv \frac{8q_1^2}{4\pi(z-1)} v_h^z$. In general the relative strengths of the different terms in the expression of temperature determine the behaviour of thermoelectric conductivity matrix with the temperature. Therefore, we consider the following three regimes of parameters. There are other possibilities, where two terms are comparable, but there it is difficult to obtain a scaling behaviour of the conductivities.

The first regime that we consider has strong momentum dissipaion compared to charge and magnetic field, which is given by, $B^2 z_h^{3(v-2)}, q_2^2 z_h^{2\xi-v-2} << k^2 z_h^{\xi-v} <<$

z_h^v. In this limit we obtain

$$
\begin{aligned}
\mathbb{K}^{mn} &\sim \frac{8\pi^2 T}{\kappa^2 k^4}[T^{\frac{2(v-\xi)}{v}}\delta_{mn} + 2q_2 B T^{\frac{4v-2\xi-4}{v}}\epsilon_{mn}],\\
\nu_{11}^{mn} &\sim \frac{8q_1^2}{\kappa^2 k^2}[T^{\frac{2v-4}{v}}\delta_{mn} + \frac{2q_2 B}{k^2}T^{\frac{4v-8}{v}}\epsilon_{mn}],\\
\nu_{12}^{mn} &\sim \frac{q_1}{\kappa^2 k^2}[8q_2\delta_{mn} + BT^{\frac{4v-6-\xi}{v}}\epsilon_{mn}]\\
\nu_{22}^{mn} &\sim \frac{1}{2\kappa^2}[T^{\frac{2v-2-\xi}{v}}\delta_{mn} + \frac{4q_2 B}{k^2}T^{\frac{4v-6-\xi}{z}}\epsilon_{mn}],\\
a_1^{mn} &\sim -\frac{8\pi q_1}{\kappa^2 k^2}[T^{\frac{2v-\xi-2}{v}}\delta_{mn} + \frac{2q_2 B}{k^2}T^{\frac{2v-4}{v}}\epsilon_{mn}],\\
a_2^{mn} &\sim -\frac{8\pi}{\kappa^2 k^2}[q_2 T^{\frac{2v-\xi-2}{v}}\delta_{mn} + \frac{B}{8}T^{\frac{4v-2\xi-4}{v}}\epsilon_{mn}].
\end{aligned}
\tag{6.4.15}
$$

The Hall angle is given approximately by $\theta_H \sim T^{\frac{2v-4}{v}}$. Since $\xi < 0$ it is not feasible to obtain linear resistivity for ν_{22}^{xx} in this regime. Choosing $v = 1$ we get $\theta_H \sim 1/T^2$ and $\nu_{22}^{xx} \sim T^{-\xi}$ shows a positive power of T for conductivity. Instead if we choose the relation, $B^2 z_h^{3(v-2)} << k^2 z_h^{\xi-v} << q_2^2 z_h^{2\xi-v-2} << z_h^v, k^2 >> 2q_2 B z_h^{2v-4}$, then except ν_{22}, all the coefficients will remain the same. It becomes

$$
\nu_{22}^{mn} = \frac{8q_2^2}{\kappa^2 k^2}[T^{\frac{2v-4}{v}}\delta_{mn} + \frac{2q_2 B}{k^2}T^{\frac{4v-8}{v}}\epsilon_{mn}]. \tag{6.4.16}
$$

We note that in this regime, ν_{22}^{xx} and Hall angle have the similar temperature dependence. So for $z = 1$ both the expressions scale as $\sim T^{-2}$. If we choose $v = 4/3$ we get $\nu_{22}^{xx} \sim T^{-1}$ implying linear resistivity. However, in this case Hall angle also becomes $\theta_H \sim T^{-1}$.

The next regime is characterised by dominance of charge over momentum dissipation and magnetic field. So the respective regime is expressed by the relation

$B^2 z_h^{3(v-2)}, k^2 z_h^{\xi-v} << q_2^2 z_h^{2\xi-v-2} << z_h^z$. The expressions of the conductivities are:

$$
\begin{aligned}
\mathbb{K}^{mn} &\sim \frac{8\pi^2 T}{\kappa^2(2q_2B)}[\frac{1}{2q_2B}T^{\frac{2(4-\xi-v)}{v}}\delta_{mn} + \frac{1}{p^2}T^{\frac{2(2-\xi)}{v}}\epsilon_{mn}], \text{for } B^2 z_h^{3(v-2)} << p^2 z_h^{\xi-v},\\
a_1^{mn} &\sim -\frac{8\pi q_1}{\kappa^2}[\frac{k^2}{4q_2^2B^2}T^{\frac{6-2v-\xi}{v}}\delta_{mn} + \frac{1}{2q_2B}T^{\frac{2-\xi}{v}}\epsilon_{mn}] \quad \text{for} \quad B^2 z_h^{3(z-2)} << p^2 z_h^{\xi-v},\\
a_1^{mn} &\sim -\frac{8\pi q_1}{\kappa^2}[\frac{4}{B^2}T^{\frac{2(v-\xi)}{v}}\delta_{mn} + \frac{1}{2q_2B}T^{\frac{2-\xi}{v}}\epsilon_{mn}] \quad \text{for} \quad k^2 z_h^{\xi-v} << B^2 z_h^{3(v-2)},\\
a_2^{mn} &\sim -\frac{8\pi}{\kappa^2}[\frac{k^2}{4q_2B^2}T^{\frac{6-2v-\xi}{v}}\delta_{mn} + \frac{1}{2B}T^{\frac{2-\xi}{v}}\epsilon_{mn}],\\
\nu_{11}^{mn} &\sim \frac{q_1^2}{2\kappa^2}[\frac{4k^2}{q_2^2B^2}T^{\frac{4-2v}{v}}\delta_{mn} + \frac{16}{2q_2B}\epsilon_{mn}], \quad \text{for} \quad k^2 z_h^{\xi} >> (q_2 B z_h^{v+\xi-4})^2,\\
&\sim \frac{q_1^2}{2\kappa^2}[T^{\frac{\xi-4}{v}}\delta_{mn} + \frac{16}{2q_2B}\epsilon_{ba}], \quad \text{for} \quad k^2 z_h^{\xi} << (q_2 B z_h^{v+\xi-4})^2,\\
\nu_{12}^{mn} &\sim \frac{8q_1q_2}{\kappa^2}[\frac{k^2}{4q_2^2B^2}T^{\frac{8-4v}{v}}\delta_{mn} + \frac{1}{2q_2B}T^{\frac{4-2v}{v}}\epsilon_{mn}],\\
\nu_{22}^{mn} &\sim \frac{1}{2\kappa^2}[\frac{4k^2}{B^2}T^{\frac{(4-2v)}{v}}\delta_{mn} + \frac{8q_2}{B}\epsilon_{mn}].
\end{aligned}
\tag{6.4.17}
$$

We note that in this regime, ν_{22}^{xx} and Hall angle have opposite temperature dependence. For $z=1$ they scale as T^2 and T^{-2} respectively with the temperature, while for $z=2$, both will be temperature independent. The third regime is dominated by the background magnetic field over the momentum dissipation and charge. In that regime, the approximate temperature dependence are $\nu_{22}^{xx} \sim T^{\frac{(4-2v)}{v}}$ with Hall angle having opposite temperature dependence.

For small temperature, we can identify the following regime of parameters. $B^2 z_h^{3(v-2)}, q_2^2 z_h^{2\xi-v-2} << k^2 z_h^{\xi-v} \lesssim z_h^v$, $B^2 z_h^{3(v-2)}, k^2 z_h^{\xi-v} << q_2^2 z_h^{2\xi-v-2} \lesssim z_h^v$ and $k^2 z_h^{\xi-v}, q_2^2 z_h^{2\xi-v-2} << B^2 z_h^{3(v-2)} \lesssim z_h^v$. However, to obtain an analytical expression for temperature for this regime is quite difficult. One can obtain the dependence on z_h from above by replacing T by z_h^v in (6.4.15) and (6.4.19) respectively in the three regimes.

For $\xi > 0$ first term is negative and so large temperature may corresponds to the region depending on whether $k^2 z_h^{\xi-v}$, $q_2^2 z_h^{2\xi-v-2}$ or $B^2 z_h^{3(v-2)}$ dominates. In these regimes, the expression of temperature can be approximated by $T \equiv \frac{k^2}{4\pi(2-\xi)} z_h^{\xi-v}$,

$T \equiv \frac{8q_2^2}{4\pi(2-\xi)} z_h^{2\xi-v-2}$ or $T \equiv \frac{B_2^2}{16\pi(2-v)} v_h^{3v-6}$, respectively. The scalings of the conductivity matrix for various regimes will be as follows:

If we consider the parameter region corresponding to strong momentum dissipation, $B^2 z_h^{3(v-2)}$, $q_2^2 z_h^{2\xi-v-2} << k^2 z_h^{\xi-v}$ we get

$$
\begin{aligned}
\mathbb{K}^{mn} &\sim \frac{8\pi^2 T}{\kappa^2 k^4}[\left(\frac{T}{k^2}\right)^{\frac{2(v-\xi)}{\xi-v}} \delta_{mn} + 2\frac{q_2 B}{k^2}\left(\frac{T}{k^2}\right)^{\frac{4v-2\xi-4}{\xi-v}} \epsilon_{mn}], \\
a_1^{mn} &\sim -\frac{8\pi q_1}{\kappa^2 k^2}[\left(\frac{T}{k^2}\right)^{\frac{2v-\xi-2}{\xi-v}} \delta_{mn} + \frac{2q_2 B}{k^2}\left(\frac{T}{k^2}\right)^{\frac{2v-4}{\xi-v}} \epsilon_{mn}], \\
a_2^{mn} &\sim -\frac{8\pi}{\kappa^2 k^2}[q_2\left(\frac{T}{k^2}\right)^{\frac{2v-\xi-2}{\xi-v}} \delta_{mn} + \frac{B}{8}\left(\frac{T}{k^2}\right)^{\frac{4v-2\xi-4}{\xi-v}} \epsilon_{mn}], \\
\nu_{11}^{mn} &\sim \frac{8q_1^2}{\kappa^2 k^2}[\left(\frac{T}{k^2}\right)^{\frac{2v-4}{\xi-v}} \delta_{mn} + \frac{2q_2 B}{k^2}\left(\frac{T}{k^2}\right)^{\frac{4v-8}{\xi-v}} \epsilon_{mn}], \\
\nu_{12}^{mn} &\sim \frac{q_1}{\kappa^2 k^2}[8q_2\delta_{mn} + B\left(\frac{T}{k^2}\right)^{\frac{2v-2-\xi}{\xi-v}} \epsilon_{mn}], \\
\nu_{22}^{mn} &\sim \frac{1}{2\kappa^2}[\left(\frac{T}{k^2}\right)^{\frac{2v-2-\xi}{\xi-v}} \delta_{mn} + \frac{4q_2 B}{k^2}\left(\frac{T}{k^2}\right)^{\frac{4v-6-\xi}{\xi-v}} \epsilon_{mn}.
\end{aligned}
\tag{6.4.18}
$$

For $v \to 2$, $\nu_{22}^{xx} \sim T^{-1}$, however the Hall angle becomes independent of temperature.

If we consider the regime, where charge is strong compared to other two factors, given by $B^2 z_h^{3(v-2)}$, $k^2 z_h^{\xi-v} << q_2^2 z_h^{2\xi-v-2}$, conductivities coming as

$$\mathbb{K}^{mn} \sim \frac{8\pi^2 T}{\kappa^2 k^2}[\frac{k^2}{4q_2^2 B^2}\left(\frac{T}{q_2^2}\right)^{\frac{8-2v-2\xi}{2\xi-v-2}}\delta_{mn} + \frac{1}{2q_2 B}\left(\frac{T}{q_2^2}\right)^{\frac{2(2-\xi)}{2\xi-v-2}}\epsilon_{mn}],$$

for $B^2 v_h^{3(v-2)} << p^2 z_h^{\xi-v}$,

$$\sim \frac{8\pi^2 T}{\kappa^2 k^2}[\frac{1}{16q_2^2}\left(\frac{T}{q_2^2}\right)^{\frac{2v+2-3\xi}{2\xi-v-2}}\delta_{mn} + \frac{1}{2q_2 B}\left(\frac{T}{q_2^2}\right)^{\frac{2(2-\xi)}{2\xi-v-2}}\epsilon_{mn}],$$

for $k^2 z_h^{\xi-v} << B^2 z_h^{3(v-2)}$,

$$a_1^{mn} \sim -\frac{8\pi q_1}{\kappa^2}[\frac{k^2}{4q_2^2 B^2}\left(\frac{T}{q_2^2}\right)^{\frac{6-2v-\xi}{2\xi-v-2}}\delta_{mn} + \frac{1}{2q_2 B}\left(\frac{T}{q_2^2}\right)^{\frac{2-\xi}{2\xi-v-2}}\epsilon_{mn}]$$

for $B^2 z_h^{3(v-2)} << k^2 z_h^{\xi-v}$,

$$a_1^{mn} \sim -\frac{8\pi q_1}{\kappa^2}[\frac{1}{16q_2^2}\left(\frac{T}{q_2^2}\right)^{\frac{2(v-\xi)}{2\xi-v-2}}\delta_{mn} + \frac{1}{2q_2 B}\left(\frac{T}{q_2^2}\right)^{\frac{2-\xi}{2\xi-v-2}}\epsilon_{mn}]$$

for $k^2 z_h^{\xi-v} << B^2 z_h^{3(v-2)}$,

$$a_2^{mn} \sim -\frac{8\pi}{\kappa^2}[\frac{p^2}{4q_2 B^2}\left(\frac{T}{q_2^2}\right)^{\frac{6-2v-\xi}{2\xi-v-2}}\delta_{mn} + \frac{1}{2B}\left(\frac{T}{q_2^2}\right)^{\frac{2-\xi}{2\xi-v-2}}\epsilon_{mn}].$$

$$\nu_{11}^{mn} \sim \frac{8q_1^2}{\kappa^2}[\frac{k^2}{4q_2^2 B^2}\left(\frac{T}{q_2^2}\right)^{\frac{4-2v}{2\xi-v-2}}\delta_{mn} + \frac{1}{2q_2 B}\epsilon_{mn}],$$

for $k^2 z_h^{\xi} >> (2q_2 B z_h^{v+\xi-4})^2$,

$$\sim \frac{1}{2\kappa^2}[\left(\frac{T}{q_2^2}\right)^{\frac{\xi-4}{2\xi-v-2}}\delta_{mn} + \frac{1}{2q_2 B}\epsilon_{mn}],$$

for $k^2 z_h^{\xi} << (2q_2 B z_h^{v+\xi-4})^2$,

$$\sim \frac{8q_1^2}{\kappa^2}[\frac{1}{16q_2^2}\left(\frac{T}{q_2^2}\right)^{\frac{2v-2-\xi}{2\xi-v-2}}\delta_{mn} + \frac{1}{2q_2 B}\epsilon_{mn}],$$

for $k^2 z_h^{\xi-v} << B^2 z_h^{3(v-2)}$,

$$\begin{aligned}\nu_{12}^{mn} &\sim \frac{8q_1}{\kappa^2}[\frac{k^2}{4q_2 B^2}\left(\frac{T}{q_2^2}\right)^{\frac{8-4v}{2\xi-v-2}}\delta_{mn} + \frac{1}{2B}\left(\frac{T}{q_2^2}\right)^{\frac{4-2v}{2\xi-v-2}}\epsilon_{mn}],\\ \nu_{22}^{mn} &\sim \frac{1}{2\kappa^2}[\frac{4k^2}{B^2}\left(\frac{T}{q_2^2}\right)^{\frac{4-2v}{2\xi-v-2}}\delta_{mn} + \frac{8q_2}{B}\epsilon_{ba}].\end{aligned} \tag{6.4.19}$$

We observed from above, ν_{22}^{xx} and Hall angle behave with temperature in opposite manner. Though for $z = 1$ one obtains $\nu_{22}^{xx} \sim T^{-1}$, but the Hall angle does not depend on temperature. One can choose the small temperature limit in a similar way as in the case of $\xi < 0$. The behaviour will be same to those obtained in the case of $\xi < 0$.

We have seen contributions of different terms in the expression of temperature determines the behaviour of the thermoelectric coefficients. For high temperature limits there are several regions in the parameter space leading to scaling with temperature, while for small temperature, it is almost impossible to identify the behaviour with specific powers of temperature. A numerical procedure may provide more precise temperature dependence.

6.5 Conclusion

To summarise, in this chapter we have analyzed thermoelectric properties of the boundary systems dual to hyperscaling violating Lifshitz geometry. In order to include the effects of magnetic fields we consider the dyonically charged black hole as the background. We have used the method given in [88]. This involves obtaining solutions of necessary fluctuations in metrics and gauge fields around the background from its linearised equation. Then thermoelectric coefficients can be obtained from the asymptotic behaviour of fluctuations in low frequency limit. This method [88] though more invovled, has the advantage that the boundary operators can be identified explicitly and different boundary conditions can be incorporated.

Here we have obtained analytic expression of various thermoelectric coefficients in terms of temperature in different limit. However, we can analytically discuss only a few specific regimes because the expression of temperature in general is too complicated,. In one of such regimes, we obtain linear temperature dependence of resistivity for $z = 4/3$, though Hall angle scales inversely with temperatrure. For $z = 1$ we find $1/T^2$ behaviour. The above result obtained for dyonic background may follow [88] from electrically charged background using mixed boundary condition on the gauge field. Further extension of this work is to study behaviour of AC conductivity with variation of temperature for intermediate frequencies, which may require a numerical procedure. As suggested in [96], additional exponents may be obtained by turning on mass for the bulk gauge field. This method can be used to analyse other models towards obtaining agreement with experimental observations.

Chapter 7

Discussion

High temperature superconductors and their close cousins remain exciting and challenging arena as the dynamics are determined by strongly correlated systems. Gauge/gravity duality is a powerful machinery that can translate the dynamics of strongly coupled field theory into weakly coupled gravity theory and in this thesis we have explored some of the aspects of the superconducting and other phases using this duality. We have adopted a bottom up approach where one constructs the gravity theory by incorporating appropriate fields and the symmetries expected in the dual field theory model. This approach is quite flexible and one can construct a tailor made gravity action in accordance with the requirement.

We begin our discussion with a gravity model, where we considered along with Einstein gravity, $SU(2) \times U(1)$ gauge theory with scalar in adjoint of the $SU(2)$. The different fields condense giving rise to different superconducting phases of the dual theory. A Chern-Simons term also has been incorporated to obtain spatial modulation for constant electric field.

Since the equations of motion are quite involved we have taken recourse to numerical computation. We have also restricted our analysis to the probe approximation, that is we have ignored the back reaction on the geometry. We find that below some critical temperature, RN AdS black hole develops instability leading to condensation of scalar and vector fields. For this model we find s-wave and p-wave phases as well as coexistence of them. Since near horizon geometry of RN AdS black hole along with some deformation corresponds to metallic phase [74], this can be interpreted as instability of the same. We have studied the free energy of the different phases as a function of temperature. We find the s-wave phase is always thermodynamically favoured below critical temperature. For a model without the scalar field and with a vector field it will be p-wave phase. From the study of the free energy we find all the phase transitions are of second order, which is consistent with the earlier results [6, 11, 39]. Due to presence of Chern-Simons term, we find helical solutions with pitch k. For the p-wave phase, we have studied thermodynamic free energy for various values of pitch (k) and find there is a critical value of pitch for which free energy becomes minimum. From the plots of free energies it turns out the phase transitions are of second order in nature.

We further extend our study to explore the change of the scenario with addition of higher derivative correction to the gravity model. We added neutral Gauss Bonnet term, as it provides black hole solution. We find higher derivative corrections tend to suppress the phase transitions. In particular, for both s wave and p wave phase the critical temperature decreases with the increase of the strength of higher derivative correction. Similar result was obtained for other models [38] . In principle, it could be that for sufficiently large coefficient of higher derivative term, critical temmperature will come down to zero. However, in our numerical method, it is difficult to study the system near zero temperature. It has been found that RN AdS black hole is also unstable to decay into AdS soliton solution [41]. Such transition occurs at zero temperature with variation of chemical potential. We have studied the effect of higher curvature on such transition and find the critical value of chemical potential increases as the coefficient of higher curvature term becomes stronger.

From the experimental study of phase structure of high temperature superconductors and related material as given in Figure.(1.1) it has been found that it admits transition between metallic and an insulating antiferromagnetic phase. This appears on variation of doping of the material. Since in the present model, RN-AdS black hole, whose near horizon along with some deformation, can be identified with metallic phase, develops instability towards a helical black hole with $SU(2)$ spontaneously broken through condensation it suggests it could have a similar phase transition. In order to explore such a phase transition in the present model we have used the techniques of holographic renormalisation group flow. In this holographic RG flow, we expect the phase to appear as fixed points of the theory, i.e. simaltaneous zeroes of the β functions. Following Hamilton Jacobi formalism we have constructed the potential equations and find the gravitational Callan Symanzik equation along with the β functions. We found that simultaneous zeros of β functions gives two fixed points. First is AdS RN black hole where the near horizon geometry along with some deformation, gives metallic nonmagnetic phase, as we mentioned. Second fixed point spontaneously breaks $SU(2) \to U(1)$ symmetry, and it may corresponds to antiferromagnetic phase [36] . To examine the insulating nature and other aspects of this phase requires further study.

Next we extend our analysis to explore anomalous transport properties in strange metal phase. New materials were discovered such as heavy fermion superconductors, cuprate high T_c superconductors which are characterised by behaviour of the transport phenomena different from that predicted by Fermi liquid theory. As suggested in [100], such a phase can be holographically obtained by considering hyperscaling violating geometry. To understand this strange metallic phase we have considered an Einstein-Maxwell-Axion-Dilaton system with $U(1) \times U(1)$ gauge fields in four dimensions. The coupling of gauge fields depend on dilaton and this system admits asymptotically Lifshitz hyperscaling violating black hole solution [88]. A background magnetic field is turned on that enables us to study magnetic properties of the boundary system as well. Instead of using near horizon analysis [87] we considered the equations of linearised fluctuations around the black hole solution [88]. Using Kubo formula, we obtain thermoelectric DC conductivity of boundary theory and Hall angles. As it has been pointed out in [88] different boundary conditions may lead to different behaviour and our method is amenable to incorporate Dirichlet, Neumann and other mixed boundary conditions. From our result, we identified several scaling regimes of transport coefficients. For different choices of parameters the behaviour varies and in particular, we find linear temperature dependence of resistivity, though hall angle turns out to be T^{-2}.

Though the study of the model with $SU(2) \times U(1)$ gauge fields provides us a fair idea of the phase structure there are a few limitations. Because of the fact that the equations involved are quite complicated we have restricted ourselves to the probe approximation and ignored back reaction of the matter on gravity as a first order approximation. Indeed, this may not be a poor approximation, in the sense that in the limit of large charge it may be a valid approximation. It is important and informative to obtain a full-fledged solution of the gravity theory coupled with matter. In fact for some simplet set-up such a back reacted black hole solutions have been obtained [25, 26, 76]. Studying the differnt phases for such a solution will give a more complete understanding. In the case of higher derivative correction for condensation of matter fields, we could not study the behaviour near zero temperature. It may be informative to obtain a full picture of the dependence on the coefficient of higher derivative term down to $T = 0$ as that will clarify whether stronger higher derivative term may lead to absence of phase transition.

In the case of renormalised group approach a more detailed study of the fixed points from thermodynamical point of view would provide a more detailed picture. In general, the actions get renormalised at the boundary, which requires introduction of necessary counterterms. A full fledged holographically renormalised action, along with the counterterms will enable us to compute two point functions at the IR and UV limit. Such an action using Kubo formula, will tell us about the behaviours of the transport coefficients as well. We have analysed transport properties for hyperscaling violating geometry but this model could not reproduce all the features of strange metallic phase. An extension of this study could be to identify appropriate gravity models which may reproduce the experimental features. It may also be interesting to explore possibility of reaching other phases using RG approach and studying transport properties of them, in particular those of antiferromagnetic phase.

Since these studies are aimed at undersatnding features of condensed matter systems, it will be more realistic to consider lattice at the boundary. There are several mechanisms to incorporate the effects of underlying lattice, which may involve choosing chemical potential or axion to be periodic. As a continuation of these studies one may consider how presence of underlying lattices may modify different aspects of the systems. To summarise the phases of superconductors and related materials provide an exciting and wide arena for studying strongly coupled systems. Methods of gauge/gravity duality are quite successful but there remains various aspects to explore.

Appendix A

Equations of motion and solution of the fields

A.1 Equations of motion

Here we partially solve the equations of motion from section 5 to write down some of the geometric parameters in terms of potentials. First we write down the equations of motion.

$$
\begin{aligned}
0 &= a''(r) + \left\{ x_1' + x_2' + x_3' + \frac{1}{2}\left(\frac{U'}{U} - \frac{g'}{g}\right)\right\} a'(r) \\
&+ e^{-x_1(r)-x_2(r)-x_3(r)}\sqrt{\frac{g}{U}}w(r)w'(r) - \left\{\frac{\phi^\dagger(r)\phi(r)}{U(r)}\right\} a(r), \\
0 &= w''(r) + \left\{ x_1' + x_2' - x_3' + \frac{1}{2}\left(\frac{U'}{U} + \frac{g'}{g}\right)\right\} w'(r) - k^2 e^{2(x_2-x_1-x_3)}\frac{w(r)}{U(r)} \\
& +k\kappa a'(r)\frac{w(r)}{U(r)}e^{-(x_1-x_2-x_3)} - \phi^\dagger\phi\frac{w(r)}{U(r)}, \\
0 &= \phi''(r) + \left\{ x_1' + x_2' + x_3' + \frac{1}{2}\left(\frac{U'}{U} + \frac{g'}{g}\right)\right\} \phi'(r) + \frac{a(r)^2}{U(r)^2}\phi(r) \\
&- m^2\phi(r) - \frac{w(r)^2}{U(r)}e^{-2x_3(r)}\phi(r), \\
0 &= x_1'' + x_2'' + x_1'^2 + x_2'^2 + x_1'x_2' + \left(\frac{1}{2}\frac{U'}{U} + \frac{1}{2}\frac{g'}{g}\right)(x_1' + x_2') \\
&- \frac{k^2}{2U}e^{-2x_1} - \frac{k^2}{4U}e^{2(x_2-x_1-x_3)} + \frac{3k^2}{4U}e^{2(x_2-x_1-x_3)} \\
& +\frac{1}{2}\left(\frac{g''(r)}{g(r)} + \frac{1}{2}\left[\frac{U'(r)g'(r)}{U(r)g(r)} - \frac{(g'(r))^2}{(g(r))^2}\right]\right) \\
&+ \frac{6}{U} + \frac{3}{8U}k^2e^{-2x_1}[w_1(r)]^2e^{-2x_3} - \frac{1}{8}e^{-2x_2}[\partial_r w_1(r)]^2 \\
& -\frac{1}{4U}w_1(r)\phi^\dagger\phi e^{-2x_2} - \frac{1}{4U}\partial_r\phi^\dagger\partial_r\phi \\
&+ \frac{[a(r)]^2}{4U(r)g(r)}\phi^\dagger\phi + \frac{1}{8}a'(r)^2\frac{1}{g(r)} + \frac{1}{2U(r)}m^2\phi^\dagger\phi, \\
0 &= x_1'' + x_3'' + x_1'^2 + x_3'^2 + x_1'x_3' + \left(\frac{1}{2}\frac{U'}{U} + \frac{1}{2}\frac{g'}{g}\right)(x_1' + x_3') - \frac{k^2}{2U}e^{-2x_1} \\
& +\frac{3k^2}{4U}e^{2(x_2-x_1-x_3)} - \frac{k^2}{4U}e^{2(x_3-x_1-x_2)} \\
&+ \frac{1}{2}\left(\frac{g''(r)}{g(r)} + \frac{1}{2}\left[\frac{U'(r)g'(r)}{U(r)g(r)} - \frac{(g'(r))^2}{(g(r))^2}\right]\right) + \frac{6}{U} \\
& +\frac{3}{8}e^{-2x_2}w'^2 - \frac{k^2}{8U}e^{-2(x_1+x_3)}w_1^2 + \frac{1}{4U}\frac{a(r)^2}{g(r)}\phi^\dagger\phi \\
&- \frac{1}{4}\partial_r\phi^\dagger\partial_r\phi + \frac{1}{4U}e^{-2x_2}w(r)^2\phi^\dagger\phi - \frac{1}{4}m^2\phi^\dagger\phi, \\
0 &= x_2'' + x_3'' + x_2'^2 + x_3'^2 + x_2'x_3' \\
&+ \left(\frac{1}{2}\frac{U'}{U} + \frac{1}{2}\frac{g'}{g}\right)(x_2' + x_3') + \frac{k^2}{2U}e^{-2x_1} - \frac{k^2}{4U}e^{2(x_2-x_1-x_3)} \\
& -\frac{k^2}{4U}e^{2(x_3-x_1-x_2)} + \frac{1}{2}\left(\frac{g''(r)}{g(r)} + \frac{1}{2}\left[\frac{U'(r)g'(r)}{U(r)g(r)} - \frac{(g'(r))^2}{(g(r))^2}\right]\right) \\
&+ \frac{6}{U} + \frac{3k^2}{8U}e^{-2(x_3+x_1)}w(r)^2 - \frac{k^2}{8}e^{-2x_2}w'^2 \\
& +\frac{1}{4}\phi'^2 - \frac{a(r)^2}{4U(r)g(r)}\phi^\dagger\phi + \frac{1}{4U}e^{-2x_2}w(r)^2(\phi^\dagger\phi) + \frac{m^2}{4U}\phi^\dagger\phi + \frac{a'^2}{8f(r)},
\end{aligned}
\tag{A.1.1}
$$

$$
\begin{aligned}
0 &= \frac{g'}{2g}(x_1'+x_2'+x_3') + (x_1'x_2'+x_2'x_3'+x_3'x_1') + \frac{k^2}{U}e^{-2x_1}Sin\,h^2(x_2-x_3) \\
&+ \frac{3}{8}e^{-2x_2}w' - \frac{k^2}{8U}e^{-2(x_1+x_3)}w^2 \\
&\quad + \frac{1}{4}\partial_r\phi^\dagger\partial_r\phi + \frac{1}{4}\frac{a(r)^2}{g(r)U(r)}\phi^\dagger\phi - \frac{1}{4U}e^{-2x_2}w^2\phi^\dagger\phi - \frac{M^2}{4U}\phi^\dagger\phi - \frac{3}{8}\frac{a'^2}{g(r)}, \\
0 &= \left[x_1''+x_2''+x_3''+x_1'^2+x_2'^2+x_3'^2\right] + \frac{U'}{2U}\left[x_1'+x_2'+x_3'\right] \\
&+ x_1'x_2'+x_2'x_3'+x_1'x_3' - \frac{1}{2U}k^2e^{-2x_1} \\
&\quad + \frac{k^2}{2U}e^{2(x_2-x_3-x_1)} + \frac{k^2}{2U}e^{2(x_3-x_2-x_1)} - \frac{1}{8}e^{-2x_2}(\partial_r w(r))^2 \\
&- \frac{k^2}{8U}e^{-2(x_1+x_3)}w^2 - \frac{1}{4gU}a^2\phi^\dagger\phi - \frac{1}{4}\partial_r\phi^\dagger\partial_r\phi \\
&\quad - \frac{1}{4U}e^{-2x_2}w^2\phi^\dagger\phi - \frac{1}{4U}m^2\phi^\dagger\phi\frac{3}{8}\frac{a'^2}{g}. \qquad \text{(A.1.2)}
\end{aligned}
$$

A.2 U(1) gauge field solution

In this section we will consider the equation of U(1) gauge field, in order to find a solution of the field in terms of the potentials, as we described. Since we obtain the expression of the potential perturbatively, so this process actually makes it feasible to obtain the expression of U(1) gauge field, near critical point. The equation of U(1) gauge field a(r) is

$$
\begin{aligned}
&a''(r) + \left\{x_1'+x_2'+x_3'+\frac{1}{2}\left(\frac{U'}{U}-\frac{g'}{g}\right)\right\}a'(r) \\
&- e^{-x_1(r)-x_2(r)-x_3(r)}\sqrt{\frac{g}{U}}w(r)w'(r) - \left\{\frac{\phi^\dagger(r)\phi(r)}{U(r)}\right\}a(r) = 0. \qquad \text{(A.2.1)}
\end{aligned}
$$

In our ansatz of the fixed point, we choose, around the fixed point the SU(2) gauge field $w(r) = 0$. In the next section, we are going to establish the fact that that near critical point $w(r)^2$ term can also be ignored. So expressing the metric-derivatives in terms of the potentials, following (5.2.27) and (5.2.28), we can express (A.2.1) as

$$
\begin{aligned}
& \frac{d}{dr}\left\{\frac{-2a(r)\phi^{\dagger}(r)\phi(r)}{g(r)}M\left(\phi^{\dagger}(r)\phi(r)\right)\right\} \\
+ \quad & \frac{W\left(\phi^{\dagger}(r)\phi(r)\right)}{2(d-1)}\left\{\frac{-2a(r)\phi^{\dagger}(r)\phi(r)M\left(\phi^{\dagger}(r)\phi(r)\right)}{g(r)}\right\} \\
- \quad & \left\{\frac{\phi^{\dagger}(r)\phi(r)}{U(r)}\right\}a(r) = 0. \qquad \text{(A.2.2)}
\end{aligned}
$$

To solve the above equation let us substitute

$$
\frac{-2a(r)\phi^{\dagger}(r)\phi(r)}{g(r)}M\left(\phi^{\dagger}(r)\phi(r)\right) = X. \qquad \text{(A.2.3)}
$$

Since exactly at the nontrivial fixed point $M\left(\phi^{\dagger}(r)\phi(r)\right)$ vanishes so we can write near the fixed point

$$
M\left(\phi^{\dagger}(r)\phi(r)\right) = (\phi^{\dagger}\phi-\alpha) + O\left((\phi^{\dagger}\phi-\alpha)^2\right)
$$

as we discussed in section 5.4 in the equation (5.4.3) Let us consider a solution $U(r) = g(r)$. We can solve the above equation(A.2.2) near nontrivial fixed point, by integrating the equation with the above prescribed value of M(5.4.3), as follows

$$
\begin{aligned}
\frac{dX}{dr} &+ \frac{W\left(\phi^{\dagger}(r)\phi(r)\right)}{2(d-1)}X - \frac{X}{2M\left(\phi^{\dagger}(r)\phi(r)\right)} = 0 \\
lnX &= \int dr'\left\{\frac{W\left(\phi^{\dagger}(r')\phi(r')\right)}{2(d-1)} - \frac{1}{2M\left(\phi^{\dagger}(r')\phi(r')\right)}\right\} \\
a(r) &= \left[\frac{-2\phi^{\dagger}(r)\phi(r)}{g(r)}M\left(\phi^{\dagger}(r)\phi(r)\right)\right]^{-1} \\
& .Exp\left[\int dr'\left\{\frac{W\left(\phi^{\dagger}(r')\phi(r')\right)}{2(d-1)} - .\frac{1}{2M\left(\phi^{\dagger}(r')\phi(r')\right)}\right\}\right]. \qquad \text{(A.2.4)}
\end{aligned}
$$

In the above equation (A.2.4) we have chosen $U(r) = g(r)$, as given in (5.2.3) Near nontrivial fixed point we can substitute the expression of $\phi(r), M\left(\phi^{\dagger}(r)\phi(r)\right)$, $W\left(\phi^{\dagger}(r)\phi(r)\right)$ to obtain the expression of U(1) gauge field $a(r)$.

A.3 Gravity equations

Here we express the gravity equations in terms of the potentials. In order to do so we use (5.2.27,5.2.28,5.2.29) to express the equations in the new form. Also in order to these equations we replace

$$\begin{aligned}
\left[\frac{dw\left(\phi^\dagger(r)\phi(r)\right)}{dr}\right]^2 &= \left[\frac{d}{d\phi^\dagger(r)}\left\{e^{-2x_2(r)}(D_i\phi(r))^\dagger(D_i\phi(r))\right\}\right]\times \\
&\quad \left[\frac{d}{d\phi(r)}\left\{e^{-2x_2(r)}(D_i\phi(r))^\dagger(D_i\phi(r))\right\}\right] \\
&= \left\{e^{-2x_2(r)}\phi^\dagger(r)\phi(r)w(r)M\left(\phi^\dagger(r)\phi(r)\right)w(r)\right\}^2 \\
\left[\frac{da\left(\phi^\dagger(r)\phi(r)\right)}{dr}\right]^2 &= \left\{\frac{2a(r)}{g(r)}\phi^\dagger(r)\phi(r)M\left(\phi^\dagger(r)\phi(r)\right)\right\}^2. \qquad \text{(A.3.1)}
\end{aligned}$$

Following (5.2.3), we write the equation of $g_{22}+g_{33}+g_{23}/Sin[px_1]Cos[px_1]$

$$\begin{aligned}
&-2U(r)\left\{\frac{2}{d-1}\frac{d}{dr}\left[W\left(\phi^\dagger(r)\phi(r)\right)\right]+\frac{3}{(d-1)^2}\left[W\left(\phi^\dagger(r)\phi(r)\right)\right]^2\right\}+k^2e^{-2x_1} \\
+&\quad \frac{k^2}{2}e^{2(x_2-x_1-x_3)}-\frac{3k^2}{2}e^{2(x_3-x_1-x_3)}-\frac{1}{d-1}U(r)\left\{\frac{d^2W\left(\phi^\dagger(r)\phi(r)\right)}{d\phi^\dagger(r)d\phi(r)}\right\}-12 \\
&-\frac{3}{4}k^2e^{-2x_1}[w(r)]^2e^{-2x_3}+U(r)e^{-2x_2(r)}\left\{e^{-2x_2(r)}\phi^\dagger(r)\phi(r)w(r)M\left(\phi^\dagger(r)\phi(r)\right)\right\}^2 \\
&+\frac{1}{2}\phi^\dagger(r)\phi(r)e^{-2x_2(r)}w(r)^2+\frac{U(r)}{2}\frac{\partial W\left(\phi^\dagger(r)\phi(r)\right)}{\partial\phi^\dagger(r)}\frac{\partial W\left(\phi^\dagger(r)\phi(r)\right)}{\partial\phi(r)} \\
-&\quad \frac{a(r)^2}{2g(r)}\left(\phi^\dagger(r)\phi(r)\right)-\frac{1}{4}\left\{\frac{2a(r)}{g(r)}\phi^\dagger(r)\phi(r)M\left(\phi^\dagger(r)\phi(r)\right)\right\}^2+\frac{1}{2}m^2\phi^\dagger(r)\phi(r) \\
=&\quad 0. \qquad \text{(A.3.2)}
\end{aligned}$$

Following (5.2.3), we write the equation of $g_{22}+g_{33}-g_{23}/Sin[px_1]Cos[px_1]$

$$
\begin{aligned}
&-2U(r)\left\{\frac{2}{d-1}\frac{d}{dr}\left[W\left(\phi^\dagger(r)\phi(r)\right)\right]+\frac{3}{(d-1)^2}\left[W\left(\phi^\dagger(r)\phi(r)\right)\right]^2\right\}+k^2e^{-2x_1}\\
- & \frac{3x^2}{2}e^{2(x_2-x_1-x_3)}+\frac{k^2}{2}e^{2(x_3-x_1-x_2)}-\frac{1}{d-1}U(r)\left\{\frac{d^2W\left(\phi^\dagger(r)\phi(r)\right)}{d\phi^\dagger(r)d\phi(r)}\right\}-12\\
&-\frac{3U(r)}{4}\left\{2e^{-2x_2(r)}\phi^\dagger(r)\phi(r)w(r)M\left(\phi^\dagger(r)\phi(r)\right)\right\}^2\\
&+\frac{1}{4}k^2e^{-2x_1(r)-2x_3(r)}w(r)^2+\frac{U(r)}{2}\frac{\partial W\left(\phi^\dagger(r)\phi(r)\right)}{\partial\phi^\dagger(r)}\frac{\partial W\left(\phi^\dagger(r)\phi(r)\right)}{\partial\phi(r)}\\
- & \frac{a(r)^2}{2g(r)}\left(\phi^\dagger(r)\phi(r)\right)\\
- & \frac{1}{4}\left\{\frac{2a(r)}{g(r)}\phi^\dagger(r)\phi(r)M\left(\phi^\dagger(r)\phi(r)\right)\right\}^2-\frac{1}{2}\phi^\dagger(r)\phi(r)e^{-2x_2(r)}w(r)^2\\
+ & \frac{1}{2}m^2\phi^\dagger(r)\phi(r)\\
= & 0. \qquad \text{(A.3.3)}
\end{aligned}
$$

The equation of motion of g_{11} component of metric is

$$
\begin{aligned}
&-2U(r)\left\{\frac{2}{d-1}\frac{d}{dr}\left[W\left(\phi^\dagger(r)\phi(r)\right)\right]+\frac{3}{(d-1)^2}\left[W\left(\phi^\dagger(r)\phi(r)\right)\right]^2\right\}-k^2e^{-2x_1}\\
+ & \frac{k^2}{2}e^{2(x_2-x_1-x_3)}+\frac{k^2}{2}e^{2(x_3-x_1-x_2)}+\frac{1}{d-1}U(r)\left\{\frac{dW\left(\phi^\dagger(r)\phi(r)\right)}{d\phi^\dagger(r)d\phi(r)}\right\}-12\\
&+\frac{U(r)}{4}\left\{e^{-2x_2(r)}\phi^\dagger(r)\phi(r)w(r)M\left(\phi^\dagger(r)\phi(r)\right)\right\}^2\\
&-\frac{3}{4}k^2e^{-2x_1(r)-2x_3(r)}w(r)^2-\frac{U(r)}{2}\frac{\partial W\left(\phi^\dagger(r)\phi(r)\right)}{\partial\phi^\dagger(r)}\frac{\partial W\left(\phi^\dagger(r)\phi(r)\right)}{\partial\phi(r)}\\
+ & \frac{a(r)^2}{2g(r)}\left(\phi^\dagger(r)\phi(r)\right)\\
- & \frac{1}{4}\left\{\frac{2a(r)}{g(r)}\phi^\dagger(r)\phi(r)M\left(\phi^\dagger(r)\phi(r)\right)\right\}^2-\frac{1}{2}\phi^\dagger(r)\phi(r)e^{-2x_2(r)}w(r)^2\\
+ & \frac{1}{2}m^2\phi^\dagger(r)\phi(r)\\
= & 0 \qquad \text{(A.3.4)}
\end{aligned}
$$

In order to find an expression of $g_{tt}=-g(r)$ at critical point with $g(r)=U(r)$, we consider the equation of motion of g_{tt} component, written in terms of the potentials, with $w(r)$ square terms are being ignored,

$$\begin{aligned}
& -\left\{\frac{3}{d-1}\frac{d}{dr}\left[W\left(\phi^\dagger(r)\phi(r)\right)\right]+\frac{3}{4(d-1)^2}\left[W\left(\phi^\dagger(r)\phi(r)\right)\right]^2\right\}-6 \\
- & \frac{1}{4}\frac{\partial W\left(\phi^\dagger(r)\phi(r)\right)}{\partial\phi^\dagger(r)}\frac{\partial W\left(\phi^\dagger(r)\phi(r)\right)}{\partial\phi(r)} \\
- & \frac{a(r)^2}{4g^2(r)}\left(\phi^\dagger(r)\phi(r)\right) \\
= & \frac{3}{8g(r)}\left\{\frac{2a(r)}{g(r)}\phi^\dagger(r)\phi(r)M\left(\phi^\dagger(r)\phi(r)\right)\right\}^2+\frac{1}{4g(r)}m^2\phi^\dagger(r)\phi(r). \quad \text{(A.3.5)}
\end{aligned}$$

Subtracting (A.3.3) from (A.3.2) gives

$$\begin{aligned}
& 2k^2\left[e^{2(x_2-x_1-x_3)}-e^{2(x_3-x_1-x_2)}\right] \\
+ & U(r)\left\{2e^{-2x_2(r)}\phi^\dagger(r)\phi(r)w(r)M\left(\phi^\dagger(r)\phi(r)\right)\right\}^2] \\
- & k^2e^{-2x_1(r)-2x_3(r)}w(r)^2 \\
+ & \frac{1}{2}\phi^\dagger(r)\phi(r)e^{-2x_2(r)}w(r)^2 \\
= & 0. \quad \text{(A.3.6)}
\end{aligned}$$

To understand the above relation (A.3.6), first we expand $w(r)$ around its fixed point value $w(\phi_*)$.

For any r, we can can express

$$\begin{aligned}
w(r) &= w(\phi_*)+\{\frac{\partial w(r)}{\partial\phi(r)}\frac{\partial\phi(r)}{\partial r}|_{\phi(r)=\phi_*}(\phi(r)-\phi_*) \\
&+ \frac{\partial w(r)}{\partial\phi^\dagger(r)}\frac{\partial\phi^\dagger(r)}{\partial r}|_{\phi(r)=\phi_*}(\phi^\dagger(r)-\phi_*^*)\} \\
&+ \ldots\ldots. \quad \text{(A.3.7)}
\end{aligned}$$

We choose at the fixed point

$$w(\phi_*)=0 \quad \text{(A.3.8)}$$

Then it is understood from (5.2.27) that also the first derivative of $w(r)$ vanishes at the critical point. So we note that $w(r)^2$ terms are of the order $[(\phi(r)^\dagger\phi(r)-\alpha)^4]$

Since we are making the study of the fixed point with potential terms at quadratic order, so we can ignore the $w(r)^2$ term in (A.3.6) and write the equation as

$$k^2\left[e^{2(x_2-x_1-x_3)}-e^{2(x_3-x_1-x_2)}\right]=0. \tag{A.3.9}$$

Finally from (A.3.9) we find near fixed point

$$x_2(r)=x_3(r). \tag{A.3.10}$$

We substract (A.3.4) from (A.3.3) to obtain

$$\begin{aligned}
- \quad & \frac{U(r)}{2}\frac{\partial W\left(\phi^\dagger(r)\phi(r)\right)}{\partial\phi^\dagger(r)}\frac{\partial W\left(\phi^\dagger(r)\phi(r)\right)}{\partial\phi(r)}+\frac{a(r)^2}{2g(r)}\left(\phi^\dagger(r)\phi(r)\right)=0\\
\left[\frac{a(r)}{g(r)}\right]^2 \quad = \quad & \frac{1}{\left(\phi^\dagger(r)\phi(r)\right)}\frac{\partial W\left(\phi^\dagger(r)\phi(r)\right)}{\partial\phi^\dagger(r)}\frac{\partial W\left(\phi^\dagger(r)\phi(r)\right)}{\partial\phi(r)}.
\end{aligned} \tag{A.3.11}$$

in the above we use $U(r)=g(r)$ to obtain $\frac{a(r)}{g(r)}$ in terms of $W(\phi^\dagger\phi)$. Finally we obtain the expression for g(r) as

$$\begin{aligned}
\frac{1}{g(r)} \quad = \quad & \left[\frac{3}{8}\left\{\frac{2a(r)}{g(r)}\phi^\dagger(r)\phi(r)M\left(\phi^\dagger(r)\phi(r)\right)\right]^2+\frac{1}{4}m^2\phi^\dagger(r)\phi(r)\right]^{-1}\\
& [-\left\{\frac{3}{d-1}\frac{d}{dr}\left[W\left(\phi^\dagger(r)\phi(r)\right)\right]+\frac{3}{4(d-1)^2}\left[W\left(\phi^\dagger(r)\phi(r)\right)\right]^2\right\}-6\\
- \quad & \frac{1}{4}\frac{\partial W\left(\phi^\dagger(r)\phi(r)\right)}{\partial\phi^\dagger(r)}\frac{\partial W\left(\phi^\dagger(r)\phi(r)\right)}{\partial\phi(r)}\\
- \quad & \frac{a(r)^2}{4g^2(r)}\left(\phi^\dagger(r)\phi(r)\right)]^{-1}
\end{aligned} \tag{A.3.12}$$

Here once again we ignore the quartic part of $w(r)$ which add higher order correction.

Appendix B

Identification of dual operators in hyperscaling violating geometry

To determine the thermoelectric DC conductivities in the method presented here in section 6 we have to identify the operators in the boundary theory with the parameters describing the asymptotic behaviour of the fluctuation of the fields which are solutions of the equation of motion. These has been discussed in details in [88] and in this appendix we include a brief review of their method. In order to describe their method, first we consider a new set of coordinates which parameterize the "dual frame", where radial coordinate is denoted by $\bar{r}$, which is related to the Einstein frame radial coordinate r with the relation given by $d\bar{r} = -sgn(\xi)e^{\frac{\xi}{2\mu}\phi}dr$. The reason for taking this dual coordinate is that it allows both positive and negative values of ξ and our UV boundary with this coordinate lies at $\bar{r} \to \infty$.

In order to identify the dual operators in the boundary theory corresponding to the fields in the bulk theory we consider [85, 88] the symplectic set of variables which consists of generalised coordinates and its canonically conjugate momenta, defined in the bulk Hamiltonian radial formalism. This enables one to find the most natural basis of symplectic variables which parametrize the space of the asymptotic solutions.

The metric in the Einstein or the dual frame can be decomposed as follows.

$ds^2 = dr^2 + \gamma_{ij}dx^i dx^j$, where $x^i = t, x^a$. In the radial Hamiltonian formalism the metric and the gauge field are being decomposed as

$$ds^2 = (N^2+N_iN^i)dr^2+2N_idrdx^i+\gamma_{ij}dx^idx^j, \quad A^I_\mu dx^\mu = A^I_r dr + A^I_i dx^i, \quad \text{(B.0.1)}$$

where N and N_i are the lapse and shift function respectively. Also γ_{ij} is the induced metric on constant r hypersurface. Similarly A_r and A_i are transverse and longitudinal components of the gauge fields to the constant r hypersurface. We also write the expression of extrinsic curvature, which is given by

$$K_{ij} = \frac{1}{2N}(\partial_r \gamma_{ij} - D_i N_j - D_j N_i), \quad \text{(B.0.2)}$$

where D_i is the covariant derivative expressed in terms of the metric γ_{ij}. Here according to our notation barred quantities will be used for dual frame and unbarred one for Einstein frame.

One can reduce the Lagrangian in the dual frame according to metric reduction (B.0.1), as obtained in [88] and the reduced lagrangian is given by

$$\begin{aligned} L_\xi = \frac{1}{2\kappa^2}\int d^3x\sqrt{-\bar{\gamma}}\bar{N}[(1+\frac{4\xi^2}{\alpha_\xi})\bar{K}^2 - \bar{K}^{ij}\bar{K}_{ij} - \frac{\alpha_\xi}{\bar{N}^2}(\partial_r\phi - \bar{N}^i\partial_i\phi - \frac{2\xi}{\alpha_\xi}\bar{N}\bar{K})^2 \\ - \frac{2}{\bar{N}^2}\Sigma^\xi_{IJ}(\phi)(F^I_{ri} - \bar{N}^k F^I_{ki})(F^{Ji}_r - \bar{N}^l F^{Ji}_l) - \frac{1}{\bar{N}^2}Z_\xi(\phi)(\partial_r\chi^a - \bar{N}^i\partial_i\chi^a)^2 \\ + R[\bar{\gamma}] - \alpha_\xi\partial_i\phi\bar{\partial}^i\phi - \Sigma^\xi_{IJ}F^I_{ij}F^{Jij} - Z_\xi\partial_i\chi^a\bar{\partial}^i\chi^a - V_\xi - 2\Box_{\bar{\gamma}}]e^{2\xi\phi}, \end{aligned} \quad \text{(B.0.3)}$$

One can obtain the canonical momenta in the dual frame from the above lagrangian as

$$\bar{\pi}^{ij} = \frac{\delta L}{\delta\dot{\bar{\gamma}}_{ij}}, \quad \bar{\pi}^i_I = \frac{\delta L}{\delta\dot{A}^I_i}, \quad \bar{\pi}_\phi = \frac{\delta L}{\delta\dot{\phi}}, \quad \bar{\pi}_{\chi^a} = \frac{\delta L}{\delta\dot{\chi}^a}, \quad \text{(B.0.4)}$$

with conjugate momenta of the non-dynamical fields, $\bar{N}$, $\bar{N}_i$ and A_r vanishing.

Expressing the canonically conjugate momenta in terms of quantities in the Einstein frame one gets

$$\bar{\pi}^{ij} = \frac{1}{2\kappa^2}\sqrt{-\gamma}e^{2\xi\phi}(K\gamma^{ij} - K^{ij}), \quad \bar{\pi}^i_I = -\frac{2}{\kappa^2}\sqrt{-\gamma}\Sigma_{IJ}\gamma^{ij}F^I_{rj},$$
$$\bar{\pi}_\phi = \frac{1}{\kappa^2}\sqrt{-\gamma}(2\xi K - \alpha\partial_r\phi), \quad \bar{\pi}_{\chi^a} = -\frac{1}{\kappa^2}\sqrt{-\gamma}Z\partial_r\chi^a. \tag{B.0.5}$$

The expressions of canonically conjugate momenta evaluated around the background in linearised order of perturbations in metric and other fields, gives the following expressions.

$$\pi^{ta} = \frac{1}{4\kappa^2}e^{2\xi\phi_B}e^{-3H}g^{-1/2}\partial_r(e^{4H}W^a_t),$$
$$\pi^a_I = -\frac{2}{\kappa^2}e^H g^{1/2}\Sigma_{IJ}(\partial_r\alpha^J_a + g^{-1}(\partial_r\alpha^J_t)W^a_t), \tag{B.0.6}$$
$$\pi_{\beta^a} = -\frac{1}{\kappa^2}e^{3H}g^{1/2}Z\partial_r\beta^a.$$

To make connection with the asymptotic expressions we will express the above equations in terms of Θ^a, α^I_a and β^a. We will consider only the terms in zeroeth order of ω. Furthermore, we will use the radial coordinate z instead of usual coordinate r. Here we substitute the background values of the fields and use $dr = -sgn(\xi)z^{-\xi/2}\mathcal{F}^{-1/2}(z)\frac{dz}{z}$ and consequently we obtain,

$$\pi^{ta} = -\frac{sgn(\xi)}{4\kappa^2}z^{\xi-v-1}\partial_z(z^{4-2\xi}(\Theta^{a(0)} + \frac{i\omega}{k}\beta^{a(0)})),$$
$$\pi^a_1 = \frac{sgn(\xi)}{2\kappa^2}[z^{v+\xi-3}F(z)\partial_z\alpha^{1(0)}_a + 4sgn(\xi)q_1(\Theta^{a(0)} + \frac{i\omega}{k}\beta^{a(0)})],$$
$$\pi^a_2 = \frac{sgn(\xi)}{2\kappa^2}[z^{3v+\xi-1}F(z)\partial_z\alpha^{2(0)}_a + 4sgn(\xi)q_2(\Theta^{a(0)} + \frac{i\omega}{k}\beta^{a(0)})], \tag{B.0.7}$$
$$\pi_{\beta^a} = \frac{i\omega}{2k\kappa^2}[-sgn(\xi)z^{5-v-\xi}\partial_z\Theta^{a(0)} - 4q_I\alpha^{I(0)}_a].$$

We substitute the expressions for the fields in small frequency limit to obtain the expressions of the canonical momenta. It has been explained in [88] the asymptotic expressions provide a map between the two sets. Here one set is given by the fluctuations, $\Theta^{a(0)}$, $\alpha^{I(0)}_a$, $\beta^{a(0)}$ along with their canonically conjugate momenta and the other set consists of the modes Θ^a_1, Θ^a_2, α^a_{a0}, C^a_I and β^a.

The set of fluctuations we have defined in section 6 should be identified with the local sources in the boundary theory but with these expressions they will become dependent of radial coordinate z. To properly identify the local sources and their dual operators one needs to consider holographic renormalisation of the action. Since the analysis we make here is very similar to [88] we refer their analysis for details. This identification involves the canonical transformation among the fluctuation modes and their canonically conjugate momenta, which can be realised by adding proper counterterms in the regularised action. This canonical transformation, in absence of the magnetic field has been described in details In [88]. They have considered the regularised on shell action for the model with the black hole solution we described here as the background. By means of addition of counterterms at the boundary the variables π^{ta}, A_1^a and π^{β^a} undergo the canonical transformations, while keeping A_2^a and its canonical conjugate momentum fixed.

As has been discussed earlier, since the effect of magnetic field appears at the linear order in frequency or higher but not in zeroeth order term, so the small frequency expansion of the fluctuations $\Theta^{a(0)}$, $\alpha_a^{I(0)}$, $\beta^{a(0)}$ remain the same as in the case of the absence of the magnetic field. However, there are differences in the expression of our blackening factor $\mathcal{F}(v)$ and so the counterterms will be changed in this case. In the case when the magnetic field is present we are assuming that we can make a similar canonical transformation by including the counterterms and obtain the transformed variables which are appropriate to make identification of the local sources and the dual operators on the boundary. A similar addition of the counterterms will give the following asymptotic expression of our transformed variables,

$$\begin{aligned} \Pi^{ta} &= -\frac{1}{4\kappa^2} z^{-2v}(\Theta_2^a + 4\mu^I C_I^a) + ..., \qquad & \mathfrak{a}_a^1 &= \alpha_{a0}^1 - \mu^1 \Theta_1 + ..., \\ \Pi_{\beta^a} &= \frac{-2i\omega}{k\kappa^2} q_I \mathfrak{a}_a^I + ..., \qquad & \mathfrak{a}_a^2 &= \alpha_{a0}^2 - \mu^2 \Theta_1 + ..., \end{aligned} \tag{B.0.8}$$

where the chemical potentials are given by

$$\mu^1 = -\frac{4sgn(\xi) q_1 z_h^{2+v-\xi}}{2+v-\xi}, \quad \mu^2 = -\frac{4sgn(\xi) q_2 z_h^{\xi-v}}{\xi - v}. \tag{B.0.9}$$

The transformed variables as we obtained above are related to our original symplectic variables through a canonical transformation. Following [88] here we identify the asymptotic expressions of these transformed variables as obtained above with the observables in the dual field theory as follows. By imposing different boundary condition one can achieve different holographically dual theory. For Dirichlet boundary condition on the field A_1^a, which requires addition of an additional boundary term to the on shell action along with the respective counterterms [88], the observables and the respective sources for energy flux are given by

$$\mathcal{E}^a = 2\lim_{\bar{r}\to\infty} e^{2v\bar{r}}\Pi^{ta} = -\frac{1}{2\kappa^2}(\Theta_2^a + 4\mu^I C_{I0}^a), \qquad \Theta_1^a = \lim_{\bar{r}\to\infty} e^{-2v\bar{r}} n_a, \tag{B.0.10}$$

respectively where $\bar{r}$ is related to r through the relation $r \sim \frac{2}{|\xi|}e^{-\frac{\xi\bar{r}}{2}}$ and n_a is the shift function in the decomposition of metric $\bar{\gamma}_{ij}$ as $\bar{\gamma}_{ij}dx^i dx^j = -(n^2 - n_a n^a)dt^2 + 2n_a dt dx^a + \sigma_{ab}dx^a dx^b$, $a, b = 1, 2$. Similarly the observable for $U(1)$ currents and pseudoscalars are expressed as follows

$$\mathcal{J}_I^a = \lim_{\bar{r}\to\infty} \Pi_I^a = -\frac{2}{\kappa^2}(C_{I0}^a - \frac{i\omega q_I}{k}\beta_0^a), \qquad \mathcal{X}_a = \lim_{\bar{r}\to\infty} \Pi_{\beta^a} = -\frac{2i\omega}{k\kappa^2} q_I a_a^I, \tag{B.0.11}$$

respectively and a_a^I is given by

$$a_a^I = \alpha_0^I - \mu^I \Theta_1^a. \tag{B.0.12}$$

Bibliography

[1] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," Int. J. Theor. Phys. **38**, 1113 (1999) [Adv. Theor. Math. Phys. **2**, 231 (1998)] doi:10.1023/A:1026654312961 [hep-th/9711200].

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge theory correlators from noncritical string theory," Phys. Lett. B **428**, 105 (1998) doi:10.1016/S0370-2693(98)00377-3 [hep-th/9802109].

[3] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2**, 253 (1998) [hep-th/9802150].

[4] S. A. Hartnoll, 'Lectures on holographic methods for condensed matter physics," Class. Quant. Grav. **26**, 224002 (2009) doi:10.1088/0264-9381/26/22/224002 [arXiv:0903.3246 [hep-th]].

[5] C. P. Herzog, "Lectures on Holographic Superfluidity and Superconductivity," J. Phys. A **42**, 343001 (2009) doi:10.1088/1751-8113/42/34/343001 [arXiv:0904.1975 [hep-th]].

[6] G. T. Horowitz, "Introduction to Holographic Superconductors," Lect. Notes Phys. **828**, 313 (2011) doi:10.1007/978-3-642-04864-7-10 [arXiv:1002.1722 [hep-th]].

[7] S. A. Hartnoll, "Horizons, holography and condensed matter," arXiv:1106.4324 [hep-th].

[8] S. Sachdev, "What can gauge-gravity duality teach us about condensed matter physics?," Ann. Rev. Condensed Matter Phys. **3**, 9 (2012) doi:10.1146/annurev-conmatphys-020911-125141 [arXiv:1108.1197 [cond-mat.str-el]].

[9] A. G. Green, "An Introduction to Gauge Gravity Duality and Its Application in Condensed Matter," Contemp. Phys. **54**, no. 1, 33 (2013) doi:10.1080/00107514.2013.779477 [arXiv:1304.5908 [cond-mat.str-el]].

[10] S. S. Gubser, "Breaking an Abelian gauge symmetry near a black hole horizon," Phys. Rev. D **78**, 065034 (2008) doi:10.1103/PhysRevD.78.065034 [arXiv:0801.2977 [hep-th]].

[11] S. A. Hartnoll, C. P. Herzog and G. T. Horowitz, "Building a Holographic Superconductor," Phys. Rev. Lett. **101**, 031601 (2008) doi:10.1103/PhysRevLett.101.031601 [arXiv:0803.3295 [hep-th]].

[12] S. A. Hartnoll, C. P. Herzog and G. T. Horowitz, "Holographic Superconductors," JHEP **0812**, 015 (2008) doi:10.1088/1126-6708/2008/12/015 [arXiv:0810.1563 [hep-th]].

[13] T. Albash and C. V. Johnson, "Vortex and Droplet Engineering in Holographic Superconductors," Phys. Rev. D **80**, 126009 (2009) doi:10.1103/PhysRevD.80.126009 [arXiv:0906.1795 [hep-th]].

[14] M. Montull, A. Pomarol and P. J. Silva, "The Holographic Superconductor Vortex," Phys. Rev. Lett. **103**, 091601 (2009) doi:10.1103/PhysRevLett.103.091601 [arXiv:0906.2396 [hep-th]].

[15] J. I. Koga, K. Maeda and K. Tomoda, "Holographic superconductor model in a spatially anisotropic background," Phys. Rev. D **89**, no. 10, 104024 (2014) doi:10.1103/PhysRevD.89.104024 [arXiv:1401.6501 [hep-th]].

[16] S. S. Gubser, "Colorful horizons with charge in anti-de Sitter space," Phys. Rev. Lett. **101**, 191601 (2008) doi:10.1103/PhysRevLett.101.191601 [arXiv:0803.3483 [hep-th]].

[17] S. S. Gubser and S. S. Pufu, "The Gravity dual of a p-wave superconductor," JHEP **0811**, 033 (2008) doi:10.1088/1126-6708/2008/11/033 [arXiv:0805.2960 [hep-th]].

[18] M. M. Roberts and S. A. Hartnoll, "Pseudogap and time reversal breaking in a holographic superconductor," JHEP **0808**, 035 (2008) doi:10.1088/1126-6708/2008/08/035 [arXiv:0805.3898 [hep-th]].

[19] F. Aprile, D. Rodriguez-Gomez and J. G. Russo, "p-wave Holographic Superconductors and five-dimensional gauged Supergravity," JHEP **1101**, 056 (2011) doi:10.1007/JHEP01(2011)056 [arXiv:1011.2172 [hep-th]].

[20] F. Benini, C. P. Herzog, R. Rahman and A. Yarom, "Gauge gravity duality for d-wave superconductors: prospects and challenges," JHEP **1011**, 137 (2010) doi:10.1007/JHEP11(2010)137 [arXiv:1007.1981 [hep-th]].

[21] S. Nakamura, H. Ooguri and C. S. Park, "Gravity Dual of Spatially Modulated Phase," Phys. Rev. D **81**, 044018 (2010) doi:10.1103/PhysRevD.81.044018 [arXiv:0911.0679 [hep-th]].

[22] H. Ooguri and C. S. Park, "Holographic End-Point of Spatially Modulated Phase Transition," Phys. Rev. D **82**, 126001 (2010) doi:10.1103/PhysRevD.82.126001 [arXiv:1007.3737 [hep-th]].

[23] H. Ooguri and C. S. Park, "Spatially Modulated Phase in Holographic Quark-Gluon Plasma," Phys. Rev. Lett. **106**, 061601 (2011) doi:10.1103/PhysRevLett.106.061601 [arXiv:1011.4144 [hep-th]].

[24] A. Donos, J. P. Gauntlett and C. Pantelidou, "Spatially modulated instabilities of magnetic black branes," JHEP **1201**, 061 (2012) doi:10.1007/JHEP01(2012)061 [arXiv:1109.0471 [hep-th]].

[25] A. Donos and J. P. Gauntlett, "Holographic helical superconductors," JHEP **1112**, 091 (2011) doi:10.1007/JHEP12(2011)091 [arXiv:1109.3866 [hep-th]].

[26] A. Donos and J. P. Gauntlett, "Helical superconducting black holes," Phys. Rev. Lett. **108**, 211601 (2012) doi:10.1103/PhysRevLett.108.211601 [arXiv:1203.0533 [hep-th]].

[27] A. Donos and J. P. Gauntlett, "Holographic striped phases," JHEP **1108**, 140 (2011) doi:10.1007/JHEP08(2011)140 [arXiv:1106.2004 [hep-th]].

[28] O. Bergman, N. Jokela, G. Lifschytz and M. Lippert, "Striped instability of a holographic Fermi-like liquid," JHEP **1110**, 034 (2011) doi:10.1007/JHEP10(2011)034 [arXiv:1106.3883 [hep-th]].

[29] L. A. Pando Zayas and D. Reichmann, "A Holographic Chiral $p_x + ip_y$ Superconductor," Phys. Rev. D **85**, 106012 (2012) doi:10.1103/PhysRevD.85.106012 [arXiv:1108.4022 [hep-th]].

[30] Z. Y. Nie, R. G. Cai, X. Gao and H. Zeng, "Competition between the s-wave and p-wave superconductivity phases in a holographic model," JHEP **1311**, 087 (2013) doi:10.1007/JHEP11(2013)087 [arXiv:1309.2204 [hep-th]].

[31] I. Amado, D. Arean, A. Jimenez-Alba, L. Melgar and I. S. Landea "Holographic s+p Superconductors," Phys. Rev. D **89**, no. 2, 026009 (2014) doi:10.1103/PhysRevD.89.026009 [arXiv:1309.5086 [hep-th]].

[32] D. Momeni, M. Raza and R. Myrzakulov, "Analytical coexistence of s-, p-, (s + p)-phases of a holographic superconductor," Int. J. Geom. Meth. Mod. Phys. **12**, no. 04, 1550048 (2015) doi:10.1142/S0219887815500486 [arXiv:1310.1735 [hep-th]].

[33] Z. Y. Nie, R. G. Cai, X. Gao, L. Li and H. Zeng, "Phase transitions in a holographic s + p model with back-reaction," Eur. Phys. J. C **75**, 559 (2015) doi:10.1140/epjc/s10052-015-3773-2 [arXiv:1501.00004 [hep-th]].

[34] G. L. Giordano and A. R. Lugo, "Holographic phase transitions from higgsed, non abelian charged black holes," JHEP **1507**, 172 (2015) doi:10.1007/JHEP07(2015)172 [arXiv:1501.04033 [hep-th]].

[35] E. Kiritsis and L. Li, "Holographic Competition of Phases and Superconductivity," JHEP **1601**, 147 (2016) doi:10.1007/JHEP01(2016)147 [arXiv:1510.00020 [cond-mat.str-el]].

[36] N. Iqbal, H. Liu, M. Mezei and Q. Si, "Quantum phase transitions in holographic models of magnetism and superconductors," Phys. Rev. D **82**, 045002 (2010) doi:10.1103/PhysRevD.82.045002 [arXiv:1003.0010 [hep-th]].

[37] L. J. Romans, "Gauged $N = 4$ Supergravities in Five-dimensions and Their Magnetovac Backgrounds," Nucl. Phys. B **267**, 433 (1986). doi:10.1016/0550-3213(86)90398-6

[38] R. Gregory, S. Kanno and J. Soda, "Holographic Superconductors with Higher Curvature Corrections," JHEP **0910**, 010 (2009) doi:10.1088/1126-6708/2009/10/010 [arXiv:0907.3203 [hep-th]].

[39] G. T. Horowitz and M. M. Roberts, "Holographic Superconductors with Various Condensates," Phys. Rev. D **78**, 126008 (2008) doi:10.1103/PhysRevD.78.126008 [arXiv:0810.1077 [hep-th]].

[40] S. Mukhopadhyay and C. Paul, "Phases in holographic helical superconductor," Int. J. Mod. Phys. A **32**, no. 13, 1750064 (2017). doi:10.1142/S0217751X17500646

[41] T. Nishioka, S. Ryu and T. Takayanagi, "Holographic Superconductor/Insulator Transition at Zero Temperature," JHEP **1003**, 131 (2010) doi:10.1007/JHEP03(2010)131 [arXiv:0911.0962 [hep-th]].

[42] R. Li, T. Zi and H. Zhang, "Holographic s + p insulator/superconductor phase transition at zero temperature," Phys. Lett. B **766**, 238 (2017). doi:10.1016/j.physletb.2017.01.018

[43] D. G. Boulware and S. Deser, “String Generated Gravity Models,” Phys. Rev. Lett. **55**, 2656 (1985). doi:10.1103/PhysRevLett.55.2656

[44] R. G. Cai, “Gauss-Bonnet black holes in AdS spaces,” Phys. Rev. D **65**, 084014 (2002) doi:10.1103/PhysRevD.65.084014 [hep-th/0109133].

[45] E. Witten, “Anti-de Sitter space, thermal phase transition, and confinement in gauge theories,” Adv. Theor. Math. Phys. **2**, 505 (1998) [hep-th/9803131].

[46] G. T. Horowitz and R. C. Myers, “The AdS / CFT correspondence and a new positive energy conjecture for general relativity,” Phys. Rev. D **59**, 026005 (1998) doi:10.1103/PhysRevD.59.026005 [hep-th/9808079].

[47] M. Ishihara, F. L. Lin and B. Ning, “Refined Holographic Entanglement Entropy for the AdS Solitons and AdS black Holes,” Nucl. Phys. B **872**, 392 (2013) doi:10.1016/j.nuclphysb.2013.04.003 [arXiv:1203.6153 [hep-th]].

[48] E. Witten, “Chiral Symmetry, the 1/n Expansion, and the SU(N) Thirring Model,” Nucl. Phys. B **14,5**, 110 (1978). doi:10.1016/0550-3213(78)90416-9

[49] Elias Kiritsis, Wenliang Li, Francesco Nitti, “Holographic RG flow and quantum effective action,” Fortsch. Phys. **62**,(2014)389-454 doi:10.1002/prop.201400007 [arXiv:1401.0888 [hep-th]].

[50] Elias Kiritsis, Francesco Nitti, Leandro Silva Pimenta, “Exotic RG flow from holography,” Fortsch. Phys. **65**,(2017)no.2, 1600120 doi:10.1002/prop.201600120 [arXiv:1611.05493 [hep-th]].

[51] Ken Kikuchi, Tadakatsu Sakai, “AdS/CFT and local renormalization group with gauge field,” PTEP 2016 **2016**,no.3, 033B02 doi:10.1093/ptep/ptw010 [arXiv:1511.00403 [hep-th]].

[52] E.J. Lindgren, Ioannis Papadimitriou, Anastasios Taliotis, Joris Vanhoof, “Holographic Hall conductivities from dyonic backgrounds,” JHEP 1507**094**2015 doi: 10.1007/JHEP07(2015)094 [arXiv:1505.04131 [hep-th]].

[53] Jan de Boer, "The Holographic Renormalization Group," Fortsch. Phys. **49**,(2001)339-358 doi: 10.1002/1521-3978(200105)49:4/6¡339::AID-PROP339¿3.0.CO;2-A [arXiv:0101026 [hep-th]].

[54] J. de Boer, E. Verlinde, H. Verlinde, "On the Holographic Renormalization Group," JHEP 0008 **003**,(2000) doi: 10.1088/1126-6708/2000/08/003 [arXiv:9912012 [hep-th]].

[55] Ioannis Papadimitriou, Kostas Skenderis, "Correlation Functions in Holographic RG Flows," JHEP 0410 **075**,(2004) doi: 10.1088/1126-6708/2004/10/075 [arXiv:0407071[hep-th]].

[56] Yu Nakayama, "Vector Beta function," Int. J . Mod . Phys . A. **28**,(2013)1350166 doi: 10.1142/S0217751X13501662 [arXiv:1310.0574[hep-th]].

[57] Jun Bourdier, Elias Kiritsis, "Holographic RG flows and nearly-marginal operators," Class. Quant . Grav . **31**,(2014)035011 doi: 10.1088/0264-9381/31/3/035011 [arXiv:1310.0858[hep-th]].

[58] Thomas Faulkner, Hong Liu, Mukund Rangamani, "Integrating out geometry: Holographic Wilsonian RG and the membrane paradigm," JHEP 1108 **051**,(2011) doi: 10.1007/JHEP08(2011)051 [arXiv:1010.4036[hep-th]].

[59] D.Z. Freedman, S.S. Gubser, K. Pilch, N.P. Warner , "Renormalization Group Flows from Holography–Supersymmetry and a c-Theorem," Adv. Theor. Math. Phys. **3**,(1999) 363-417 doi: 10.4310/ATMP.1999.v3.n2.a7 [arXiv:9904017[hep-th]].

[60] Idse Heemskerk, Joseph Polchinski, "Holographic and Wilsonian Renormalization Groups," JHEP 1106 **031**,(2011) doi: 10.1007/JHEP06(2011)031 [arXiv:1010.1264[hep-th]].

[61] Ioannis Papadimitriou, “Holographic Renormalization of general dilaton-axion gravity,” JHEP 1108 **119**,(2011) doi: 10.1007/JHEP08(2011)119 [arXiv:1106.4826[hep-th]].

[62] Elias Kiritsis, Vasilis Niarchos, “The holographic quantum effective potential at finite temperature and density,” JHEP 1208 **164**,(2012) doi: 10.1007/JHEP08(2012)164 [arXiv:1205.6205[hep-th]].

[63] Thomas L Curtright, Xiang Jin, Cosmas K Zachos, “RG flows, cycles, and c-theorem folklore,” Phys. Rev. Lett. **108**,(2012)131601 doi: 10.1103/PhysRevLett.108.131601 [arXiv:1111.2649[hep-th]].

[64] Sung-Sik Lee, “Quantum Renormalization Group and Holography,” JHEP 01 (2014) **076**, doi: 10.1007/JHEP01(2014)076 [arXiv:1305.3908[hep-th]].

[65] Kostas Skenderis, “Lecture Notes on Holographic Renormalization,” Class. Quant . Grav . **19**,(2002)5849-5876 doi: 10.1088/0264-9381/19/22/306 [arXiv:0209067[hep-th]].

[66] Vijay Balasubramanian, Per Kraus, “Spacetime and the Holographic Renormalization Group,” Phys. Rev. Lett. **83**,(1999)3605-3608 doi: 10.1103/PhysRevLett.83.3605 [arXiv:9903190[hep-th]].

[67] Massimo Bianchi, Daniel Z. Freedman, Kostas Skenderis, “How to go with an RG Flow,” JHEP 0108 (2014) **076**, doi: 10.1007/JHEP01(2001)041 [arXiv:0105276[hep-th]].

[68] Thomas Hertog, “Towards a Novel no-hair Theorem for Black Holes,” Phys. Rev. D. **74**,(2006) 084008 doi: 10.1103/PhysRevD.74.084008 [arXiv:0608075[gr-qc]].

[69] Gary T. Horowitz, Matthew M. Roberts, “Zero Temperature Limit of Holographic Superconductors,” JHEP 0911 (2009) **015**, doi: 10.1088/1126-6708/2009/11/015 [arXiv:0908.3677[hep-th]].

[70] Thomas Faulkner, Gary T. Horowitz, John McGreevy, Matthew M. Roberts, David Vegh, “Photoemission ”experiments” on holographic superconductors,” JHEP 1003 (2010) **121**, doi: 10.1007/JHEP03(2010)121 [arXiv:0911.3402[hep-th]].

[71] Steven S. Gubser, Abhinav Nellore, “Ground states of holographic superconductors,” Phys. Rev. D. **80**,(2009) 105007 doi:10.1103/PhysRevD.80.105007 [arXiv:0908.1972[hep-th]].

[72] Frederik Denef, Sean A. Hartnoll, Subir Sachdev, “Quantum oscillations and black hole ringing,” Phys. Rev. D. **80**,(2009) 126016 doi:10.1103/PhysRevD.80.126016 [arXiv:0908.1788[hep-th]].

[73] Steven S. Gubser, Christopher P. Herzog, Silviu S. Pufu, Tiberiu Tesileanu, “Superconductors from Superstrings,” Phys. Rev. Lett. **103**,(2009)141601 doi: 10.1103/PhysRevLett.103.141601 [arXiv:0907.3510[hep-th]].

[74] Aristomenis Donos, Sean A. Hartnoll, “Metal-insulator transition in holography,” Nature Phys. **9**,(2013)649-655 doi: 10.1038/nphys2701 [arXiv:1212.2998[hep-th]].

[75] Aristomenis Donos, Blaise Goutraux, Elias Kiritsis, “Holographic Metals and Insulators with Helical Symmetry,” JHEP 1409 (2014) **038**, doi: 10.1007/JHEP09(2014)038 [arXiv:1406.6351[hep-th]].

[76] Aristomenis Donos, Jerome P. Gauntlett, “Black holes dual to helical current phases,” Phys. Rev. D. **86**,(2012)064010 doi:10.1103/PhysRevD.86.064010 [arXiv:1204.1734[hep-th]].

[77] S. A. Hartnoll, A. Lucas and S. Sachdev, “Holographic quantum matter,” arXiv:1612.07324 [hep-th].

[78] S. Kachru, X. Liu and M. Mulligan, “Gravity duals of Lifshitz-like fixed points,” Phys. Rev. D **78**, 106005 (2008) doi:10.1103/PhysRevD.78.106005 [arXiv:0808.1725 [hep-th]].

[79] M. Taylor, "Non-relativistic holography," arXiv:0812.0530 [hep-th].

[80] S. F. Ross and O. Saremi, "Holographic stress tensor for non-relativistic theories," JHEP **0909**, 009 (2009) doi:10.1088/1126-6708/2009/09/009 [arXiv:0907.1846 [hep-th]].

[81] S. F. Ross, "Holography for asymptotically locally Lifshitz spacetimes," Class. Quant. Grav. **28**, 215019 (2011) doi:10.1088/0264-9381/28/21/215019 [arXiv:1107.4451 [hep-th]].

[82] R. B. Mann and R. McNees, "Holographic Renormalization for Asymptotically Lifshitz Spacetimes," JHEP **1110**, 129 (2011) doi:10.1007/JHEP10(2011)129 [arXiv:1107.5792 [hep-th]].

[83] M. Baggio, J. de Boer and K. Holsheimer, "Hamilton-Jacobi Renormalization for Lifshitz Spacetime," JHEP **1201**, 058 (2012) doi:10.1007/JHEP01(2012)058 [arXiv:1107.5562 [hep-th]].

[84] T. Griffin, P. Horava and C. M. Melby-Thompson, "Conformal Lifshitz Gravity from Holography," JHEP **1205**, 010 (2012) doi:10.1007/JHEP05(2012)010 [arXiv:1112.5660 [hep-th]].

[85] W. Chemissany and I. Papadimitriou, "Lifshitz holography: The whole shebang," JHEP **1501**, 052 (2015) doi:10.1007/JHEP01(2015)052 [arXiv:1408.0795 [hep-th]].

[86] S. Cremonini, H. S. Liu, H. Lu and C. N. Pope, "DC Conductivities from Non-Relativistic Scaling Geometries with Momentum Dissipation," JHEP **1704**, 009 (2017) doi:10.1007/JHEP04(2017)009 [arXiv:1608.04394 [hep-th]].

[87] N. Bhatnagar and S. Siwach, "DC conductivity with external magnetic field in hyperscaling violating geometry," Int. J. Mod. Phys. A **33**, no. 04, 1850028 (2018) doi:10.1142/S0217751X18500288 [arXiv:1707.04013 [hep-th]].

[88] S. Cremonini, M. Cvetic and I. Papadimitriou, "Thermoelectric DC conductivities in hyperscaling violating Lifshitz theories," JHEP **1804**, 099 (2018) doi:10.1007/JHEP04(2018)099 [arXiv:1801.04284 [hep-th]].

[89] X. H. Ge, Y. Tian, S. Y. Wu, S. F. Wu and S. F. Wu, "Linear and quadratic in temperature resistivity from holography," JHEP **1611**, 128 (2016) doi:10.1007/JHEP11(2016)128 [arXiv:1606.07905 [hep-th]].

[90] Z. N. Chen, X. H. Ge, S. Y. Wu, G. H. Yang and H. S. Zhang, "Magnetothermoelectric DC conductivities from holography models with hyperscaling factor in Lifshitz spacetime," Nucl. Phys. B **924**, 387 (2017) doi:10.1016/j.nuclphysb.2017.09.016 [arXiv:1709.08428 [hep-th]].

[91] A. Donos and J. P. Gauntlett, "Holographic Q-lattices," JHEP **1404**, 040 (2014) doi:10.1007/JHEP04(2014)040 [arXiv:1311.3292 [hep-th]].

[92] T. Andrade and B. Withers, "A simple holographic model of momentum relaxation," JHEP **1405**, 101 (2014) doi:10.1007/JHEP05(2014)101 [arXiv:1311.5157 [hep-th]].

[93] M. M. Caldarelli, A. Christodoulou, I. Papadimitriou and K. Skenderis, "Phases of planar AdS black holes with axionic charge," JHEP **1704**, 001 (2017) doi:10.1007/JHEP04(2017)001 [arXiv:1612.07214 [hep-th]].

[94] A. Donos and J. P. Gauntlett, "Thermoelectric DC conductivities from black hole horizons," JHEP **1411**, 081 (2014) doi:10.1007/JHEP11(2014)081 [arXiv:1406.4742 [hep-th]].

[95] M. Blake and A. Donos, "Quantum Critical Transport and the Hall Angle," Phys. Rev. Lett. **114**, no. 2, 021601 (2015) doi:10.1103/PhysRevLett.114.021601 [arXiv:1406.1659 [hep-th]].

[96] B. Gouteraux and E. Kiritsis, "Quantum critical lines in holographic phases with (un)broken symmetry," JHEP **1304**, 053 (2013) doi:10.1007/JHEP04(2013)053 [arXiv:1212.2625 [hep-th]].

[97] ’t Hooft, “A Planar Diagram Theory for Strong Interactions,” Nucl PhyB72(1974)461

[98] O.Aharony,S. Gubser,J Maldacena,H Ooguri, Y Oz, ”Large N Field Theories, String Theory and Gravity” Phys.Rept. **323**, (2000), 183-386 [arXiv:9905111 [hep-th]]

[99] Niko Jokela, Matti Jarvinen, Matthew Lippert ”Gravity dual of spin and charge density waves” JHEP **1412** (2014) 083 [arXiv:1408.1397 [hep-th]]

[100] Andreas Karch ”Conductivities for Hyperscaling Violating Geometries” JHEP **1406** (2014) 140 [arXiv:1405.2926 [hep-th]]

[101] J. Polchinski, “String theory. Vol. 1: An introduction to the bosonic string,” doi:10.1017/CBO9780511816079

[102] J. Polchinski, “String theory. Vol. 2: Superstring theory and beyond,” doi:10.1017/CBO9780511618123

[103] J. Dai, R. G. Leigh, and J. Polchinski, Mod . Phys . Lett. A **4**,(2073)1989, doi: 10.1007/978-3-642-04864-7 10 .

[104] E. Witten,”Bound States of Strings and p-Branes, ” Nucl. Phy. B **460**,(1996)335–460 , [arXiv:hep-th/ 9510135].

[105] J. D. Bekenstein,”Black holes and entropy, ” Phys.Rev.D **7**,(1973)2333–2346

[106] S. W. Hawking,” Particle Creation by Black Holes,” Commun.Math.Phys **43**,(1975)199–220

[107] S. W. Hawking and D. N. Page,” Thermodynamics of Black Holes in anti-De Sitter Space,”Commun.Math.Phys **87**,(1983)577

[108] V. L. Ginzburg and L. D. Landau,”On the Theory of superconductivity ,” Zh. Eksp. Teor. Fiz. **20** ,(1950)1064

[109] J. Bardeen, L. N. Cooper and J. R. Schrieffer,"Theory Of Superconductivity," Phys. Rev. **108** ,(1957)1175

[110] P. Breitenlohner and D. Z. Freedman,"Stability In Gauged Extended Supergravity," Annals Phys. **144** ,(1982)249

[111] I. R. Klebanov and E. Witten, "AdS/CFT correspondence and symmetry breaking," Nucl. Phy. B **556**,(1999) 89 , [arXiv:hep-th/ 9905104].

[112] M. P. Ryan and L. C. Shepley, "Homogeneous Relativistic Cosmologies," Princeton Series in Physics, ISBN 0-691-0153-0.

[113] G. T. Horowitz and A. Strominger, "Black strings and P-branes," Nucl. Phy. B **360**,(1991) 197209

[114] Aristomenis Donos, Jerome P. Gauntlett, "Navier-Stokes Equations on Black Hole Horizons and DC Thermoelectric Conductivity," Phys.Rev. D92 (2015) **12**, 121901 doi: 10.1103/PhysRevD.92.121901 [arXiv:1506.01360 [hep-th]].

[115] Elliot Banks, Aristomenis Donos, Jerome P. Gauntlett, "Thermoelectric DC conductivities and Stokes flows on black hole horizons," JHEP **1510**, 103 (2015) doi: 10.1007/JHEP10(2015)103 [arXiv:1507.00234 [hep-th]].

[116] Sean A. Hartnoll, Diego M. Hofman, "Locally critical umklapp scattering and holography," Phys.Rev.Lett. **108** (2012) 241601 doi: 10.1103/PhysRevLett.108.241601 [arXiv: 1201.3917[hep-th]].

[117] H. Nastase, "Introduction to the ADS/CFT Correspondence," Cambridge University Press,U.K,2015